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Cartesian Mind and Its Concept of Space:
A Contribution to the Project of Jacob Klein

A DISSERTATION

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By

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Building upon the arguments of Jacob Klein and Edmund Husserl that we can only understand the meaning of modern science by investigating its historical development, this dissertation aims to uncover the nature and conceptual presuppositions of Descartes’s concept of space. In analyzing this concept, the dissertation extends Klein’s analysis of Descartes’s contribution to the development of modern symbolic mathematics to Descartes’s equally important role in developing the conceptual underpinnings of modern mathematical physics by showing that Descartes’s concept of space, which spans and unites the mathematical and physical domains, is an expansion of the symbolic concept of number. The analysis of Descartes’s concept of space depends upon connecting the following aspects of Descartes’s writings: the account of mind and mathematical cognition in the Rules for the Direction of the Mind, the conceptual structure of the mathematical objects and the mathematical understanding of space in the Geometry, and the physical understanding of space in the Principles of Philosophy. The dissertation ultimately concludes that Descartes’s concept of space allows the distinctive conceptual structure of modern mathematics to be applied to the physical world, whereby that concept provides a conceptual framework within which mathematical physics can exist.
This dissertation by Andrew Joseph Romiti fulfills the dissertation requirement for the doctoral degree in Philosophy approved by Richard F. Hassing, Ph.D., as Director, and by John C. McCarthy, Ph.D., and Burt C. Hopkins, Ph.D. as Readers.

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“Only an understanding from within of the movement of modern philosophy from Descartes to the present, which is coherent despite all its contradictions, makes possible an understanding of the present itself.”

—Edmund Husserl, *The Crisis of the European Sciences and Transcendental Phenomenology*

“Descartes’s great idea . . . consists of identifying, by means of ‘methodological’ considerations, the ‘general’ object of [algebra]—which can be represented and conceived only *symbolically*—with the ‘substance’ of the world, with corporeality as ‘extensio.’ Only by virtue of this identification did symbolic mathematics gain that fundamental position in the system of knowledge which it has never since lost.”

—Jacob Klein, *Greek Mathematical Thought and the Origin of Algebra*

“The whole of my physics is nothing other than geometry.”

—René Descartes, *Letter to Mersenne, 27 July 1638*
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INTRODUCTION

The goal of this dissertation is to contribute to Jacob Klein’s project of clarifying the conceptual structure of modern scientific thought by examining a key concept in the historical development of modern mathematical science, namely, Descartes’s concept of space, which spans and unites the mathematical and physical realms. It will ultimately be shown that this concept allows the distinctive conceptual structure of modern mathematics to be applied to the physical world, whereby it provides a conceptual framework for the mathematical physics that Descartes aims to create. An investigation of this concept is therefore aimed at uncovering the conceptual presuppositions that underlie the combination of mathematics and physics that is a hallmark of modern science.

The following is divided into five chapters. Chapter 1 lays out the framework within which this study takes place by discussing the thought of Jacob Klein, who provides the starting point for the line of investigation taken up in this dissertation. Chapter 2 expands upon Klein’s analysis of Descartes’s contribution to the invention of modern symbolic mathematics by spelling out the account of mathematical cognition contained in the Rules for the Direction of the Mind, while also clarifying the understanding of the mind-world relationship that is built into that cognition. Chapter 3 extends the investigation beyond Klein’s analysis by showing that the mathematical cognition of the Rules underlies both the mathematics of the Geometry and the concept of space found in that work. Chapter 4 then examines the physical understanding of space found in the Principles of Philosophy and shows that it is related to the Geometry’s mathematical understanding of space in such a way that allows the mathematical and the physical realms to be brought together. Finally, Chapter 5 draws the major conclusions of this
study and highlights the implications of these conclusions for the understanding of Klein, Descartes, and the later development of modern science.
CHAPTER 1

Jacob Klein and the Context of the Present Study

§ 1. Introduction to Klein and His Work

In *Greek Mathematical Thought and the Origin of Algebra*, Jacob Klein sets out to reveal the fundamental difference between the ancient and modern concepts of number for the sake of clarifying the nature of the formal mathematical language that underlies modern mathematical physics. By the end of his analysis, he has shown that the invention of modern symbolic mathematics reveals a difference in the way the concepts of ancient and modern science are formed and used, and, as explicated in the following, that this difference involves different relations between the mind and the world. These results, he believes, have widespread implications for the nature of modern science as a whole and, through its influence, for modern thought as such. The importance of Klein’s work thus lies both in the depths of its analysis and the wide range of its implications.

While Klein’s work is most obviously one of historical-mathematical significance, it’s broader implications can be seen from his goal of contributing to the investigation of the nature and influence of modern science, particularly in its mathematical form, which he takes to be in need of conceptual clarification. Klein is in fact largely motivated by the conceptual difficulties modern physics was facing in his own day, identifying them as the “ultimate theme” that shapes his inquiry.¹ Before one can deal with the problems of mathematical physics, however, Klein

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believes we must “set ourselves the task of inquiring into the origin and the conceptual structure of [its] formal language” because the meaning of this science is inextricable from its mathematical form. So long as the nature of that mathematical form is not clearly understood, the conceptual structure of the sciences built upon it are also in need of clarification.

This need is all the more pressing given the role modern science plays in shaping the everyday understanding of modern man. In his lecture “The World of Physics and the ‘Natural’ World,” Klein describes the situation as follows:

mathematical physics is at the foundation of our mental and spiritual life . . . we see the world and ourselves in this world at first quite ingenuously as mathematical physics has taught us to see it, . . . the direction, the very manner of our questioning is fixed in advance by mathematical physics, and . . . even a critical attitude towards mathematical physics does not free us from its dominion.

The conceptual clarification of modern mathematics and physics is therefore of great importance, but any attempt at such clarification, according to Klein, will be seriously hampered by the divide that exists between physics and philosophy, which is itself partially a product of the “formalistic thicket” that mathematical physics depends upon. To get around this problem, he argues, we must uncover the nature of the formalism at work in modern mathematics and thus at the heart of modern physics, and for that we must turn to an historical investigation of the rise of that formal language.

Yet is there reason to think that we do not understand the nature of the formal language of modern mathematics? Klein shows that there is by revealing that we do not understand key

(Part I), and no. 2 (1936), 122-235 (Part II). For the purpose of citations, this book will be abbreviated as “GMTOA.” with page references referring to the English translation (the German original, appearing as it did at a time that could not have been more inauspicious, was not widely read and is nowadays rarely cited).

Ibid.


4 Ibid., 2.
differences in ancient versus modern mathematics, including important differences that concern formalism itself. This can be seen from the fact that the nature of ancient mathematics is not adequately understood because it is not considered first in its own terms. The loss of the self-understanding of ancient mathematics, Klein shows, ultimately stems from the fact that the conceptual presuppositions built into the nature of modern mathematics at its origin—including those connected with the rise of formalism itself—are now accepted unawares with the receipt of mathematics in its current form. Through the course of its historical development modern mathematics has become blind to its own conceptual presuppositions, and as a result it presumes them to be true of all mathematical thought. By uncovering important differences in the ancient and modern concepts of number, Klein establishes that such presuppositions do not in fact govern the conceptual structure of all mathematics. His recovery of the lost ancient mode, along with the accompanying revelation of its differences from mathematics in its modern form, thus allows Klein to highlight the conceptual structure of modern mathematics anew and thereby allow for its conceptual clarification, as well as the clarification of the broader conceptual presuppositions that turn out to have been ingested along with it.

What then is the difference between the ancient and modern concepts of number? In brief, Klein shows that the ancient concept (the Greek *arithmos* concept) is restricted to “a definite number of definite objects.” This is ultimately the case because the ancient understanding of number is dependent upon the direct apprehension of a multitude, whether that be made up of material objects perceived by the senses or purely intelligible ones apprehended by the mind. The ancient concept of number thus has nothing formal about it, nor is there any

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5 Klein, *GMTOA*, 7. See Section 6 of *Greek Mathematical Thought and the Origin of Algebra* for Klein’s demonstration of this.
formalism at work in its expression. The modern concept of number, on the other hand, is inherently formal in that it has a level of generality and indeterminacy built into it. As we will consider in detail below, Klein’s analysis shows that the modern concept of number designates multitudinousness-as-such (i.e., the general concept of number), which is only grasped indirectly by the mind’s reflecting on its own apprehension of a quantity.\(^6\) The manifestation of this apprehension, however, is dependant upon algebraic letter signs, and in this way the modern concept of number is fundamentally symbolic. The invention of algebraic symbols is therefore concomitant with the invention of the formal language that underlies modern mathematics, both of which are completely foreign to ancient mathematics.

Later in this chapter, we will spell out Klein’s accomplishment in much greater detail. First, however, we must examine the Husserlian background of Klein’s work, as that will provide us with the broader philosophical context in which Klein’s historical-conceptual achievements can best be understood. This, in turn, will prepare our discussion of the various features of Klein’s thought that are relevant to the analysis of Descartes’s concept of space in the ensuing chapters.

§ 2. Burt Hopkins on Klein’s Relation to Husserl

To properly understand the achievement of Klein’s work, it must be viewed in relation to the late thought of Edmund Husserl. This has been ably demonstrated by Burt Hopkins in his book *The Origin of the Logic of Symbolic Mathematics: Edmund Husserl and Jacob Klein*,

\(^6\) While Klein’s argument for this is contained throughout Sections 11 and 12 of *Greek Mathematical Thought and the Origin of Algebra*, see in particular pages 200-08.
wherein Hopkins thoroughly explicates the reciprocal relationship between these two thinkers. Let us rehearse the major conclusions of Hopkins’ book as that will help illuminate the significance of Klein’s work while also contributing to the framing of the present study.

Hopkins shows that Klein’s work is best understood as situated in a Husserlian context because of the two thinkers’ shared goal of investigating the historical significance of the invention of modern mathematics for the sake of understanding the nature and influence of modern mathematical science. While Husserl and Klein ultimately have different accounts of the origin of the formalization that underlies modern mathematics, Hopkins shows that their accomplishments are complementary in that Husserl provides the philosophical justification for the kind of historical investigation performed by Klein, while it is Klein’s account that actually uncovers the nature of the formalization that lies at the origin of modern mathematics.

Hopkins identifies the subject matter of his book as “the view that formalization is the fulcrum for an unprecedented transformation in how the science of the so-called West forms its concepts, a transformation that is as all-encompassing as it is invisible to this day.” According to Hopkins, the historical investigations of Klein, who is identified as the first major proponent of this view, ultimately show that the transformation in the concept of number underlying the invention of modern algebra was the first step in a larger transition whereby the understanding of what things are came to be merely conceptual and linguistic.

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8 Hopkins’s fuller articulation of this claim is as follows: “Prior to [the transformation in the concept of number], things were apprehended ‘directly,’ first through the senses and then through the employment of concepts that were apprehended as different from both the things whose apprehension they permitted and from the language that made use of concepts in order to bring about this apprehension. Subsequent to that transformation, things are apprehended ‘indirectly,’ through the mediation of both the concepts and language that now ‘define’ them. In other words, Klein’s thesis is that before that historical transformation, knowledge of the ‘being’—however mysterious or
Klein, however, does not provide a full philosophical rationale for his particular historical approach. This is where Husserl is helpful, as he attempts to give an account of the connection between the historical and epistemological investigations necessary to understand the formalization that underlies modern science. In his late works, most notably *The Crisis of European Sciences and Transcendental Phenomenology* and “The Origin of Geometry,” Husserl proposes a kind of historical reflection that aims at clarifying the genuine meaning of the exact sciences. It does so by seeking to reawaken the original evidence that led to their original establishment. This evidence, he believes, has been covered over by “sedimentation” that obscures but does not destroy the originary meanings at work in these sciences. Hopkins describes the situation thus:

Husserl worked out the nature of the method required to undertake these “epistemological-historical” investigations, which he characterizes as a “zigzag” reflection that moves from the present meaning of a science to historically prior meanings, then back again to the present meaning, all with the goal of “reactivating” in the prior meanings evidence that anticipated the contemporary meaning. He also worked out the philosophical basis of these investigations.”

This method, then, is aimed at uncovering or “de-sedimenting” the original meanings of the idealized concepts at work in the sciences for the sake of bringing their present-day meanings into clearer focus.

unknown—of things was incapable, in principle, of being identified with the concepts and language employed to apprehend them, while, subsequent to it, knowledge of the ‘being’ of things—again, however mysterious or unknown—is approached only through the concepts and language used to apprehend them.” (Ibid., 4.)

Ibid., 7. Husserl’s account of this will be discussed in detail in the following section.

For a discussion of the term “desedimentation,” see Hopkins, *Origin*, 71-72 n. 4, where he distinguishes the “desedimentation” and “reactivation” involved in the process of investigating the meaning and history of a science as follows: “the term ‘desedimentation’ is used to refer to the methodological uncovering of the sedimented history of meaning inseparable from the origin of the scientific concepts that compose the origin of a science, while ‘reactivation’ is used for the reawakening of the original evidence in which original meanings of these concepts are ‘anticipated.’”
While Klein’s work is aimed at just such an uncovering, the precise relationship between the work of Klein and Husserl is nevertheless difficult to pin down. This is due in part to Klein’s own confusing presentation of the matter. To begin with, Klein’s historical study in fact predates Husserl’s own late turn to history, the former being completed by 1933 while the latter only begins in 1934. This, however, does not stop Klein from later and retrospectively situating his own work in the context of Husserl’s late thought in his essay “Phenomenology and the History of Science,” which was published in 1940. Yet Klein never explicitly refers the one to the other. That is, in discussing Husserl’s late turn to history Klein does not mention his own work, despite articulating the need for the very task accomplished by that work; nor in his own work does he mention Husserl. In investigating this “scholarly curiosity,” Hopkins shows that despite the connections, there are important differences in the accounts of these two thinkers.

The most notable difference in Klein’s and Husserl’s accounts concerns the mathematical concept of unity. Hopkins shows that Husserl has a formalized concept of unity, which he presupposes to be obtainable by lifting off the unity of material individuals, a process that gives rise to the empty concept of the “anything-whatever” (etwas überhaupt) by leaving behind all

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11 For a full account of the relevant dates, see Hopkins, Origin, 3-8, 16 n. 5, 62 n. 54.
12 This essay was originally published in a commemorative volume published shortly after Husserl’s death in 1938 entitled Philosophical Essays in Memory of Edmund Husserl, ed. Marvin Faber (Cambridge: Harvard University Press, 1940), 143-63. It is reprinted in Jacob Klein, Lectures and Essays, ed. Robert B. Williamson and Elliott Zuckerman (Annapolis: St. John’s College Press, 1985), 65-84. (Citations to this work will refer to the latter). According to Hopkins this essay is not only one of the earliest interpretations of Husserl’s turn to history, being published within just a few years of this turn, it is also one of the best in that Klein “alone among Husserl’s commentators” has understood the importance and role of history in Husserl’s phenomenology (see Hopkins, Origin, 26, 43 & 67).
13 This phrase is originally used by Hiram Caton to describe the absence of any mention of Husserl in Klein’s book; see Hiram Caton, “Review of Jacob Klein’s Greek Mathematical Thought and the Origin of Algebra,” Studi Internazionali di Filosofia 3 (1971), 225. Hopkins, thematizing this phrase, uses this “curious” fact to set up much of the analysis of his book.
In this way, Husserl locates the origin of the formalized concepts of modern mathematics in the mind’s apprehension of objects of the world, yet he never gives an account of how this is possible.

Klein, on the other hand, locates this origin in the historical creation of the modern concept of number and its corresponding new concept of the unit. Moreover, Klein’s historical investigation of the origin of formalization in mathematics shows that formalized concepts, preeminently “anything-whatever,” cannot be obtained by merely disregarding the sensible content of an object and lifting off its unity, as presupposed by Husserl. That is to say, lurking in Husserl’s own thinking is an unquestioned supposition about the nature of the abstraction at work in formalization, which precludes him from uncovering the true nature of that formalization.

The result of Klein’s historical account thus points as a first step to a “fundamental critique” of Husserl’s attempt at uncovering the nature of formalization. The nub of this critique comes in showing that the origin of formalized concepts lies not “in the immediate experience and intuition of individual objects,” but rather “occurs at a higher conceptual level.”

Klein shows just that by revealing the nature of the “formalizing abstraction” that is responsible for the creation of formalized number. While the exact details of this abstraction will be discussed later (§§ 6-7), it is essentially characterized by the mind’s fixing its cognitive regard upon its own act of knowing, rather than directly upon individual objects. In this way, formalizing abstraction gives rise to a general and indeterminate object with no material content. Such an object, however, is dependant upon an algebraic letter sign to be grasped, as it has no determinacy itself.

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15 Ibid., 74.
and thus cannot be presented as an object without being presented by something determinate. This proves, as Hopkins puts it, that “the genuine discovery of the formal [is] dependant on the representation by letter signs of the indeterminate object of pure algebra.”

By revealing the formalizing abstraction at the heart of formalization, Klein’s account goes deeper than Husserl’s and shows its shortcomings by getting to the true root of the matter. Thus, as Hopkins shows, Klein’s work reveals that Husserl’s own attempt at uncovering the origin of formalization leaves unexamined a presupposition regarding the source of that very formalization. Yet, in uncovering something that was itself presupposed by Husserl’s own thinking, Klein’s accomplishments remain on Husserlian grounds in that it is Husserl who laid out the philosophical basis for his very approach. Thus, while Klein ultimately demonstrates the falsity of Husserl’s account of formalization, that demonstration itself turns out to be most compelling in light of Husserl’s articulation of the methodology of desedimentation. In this way, Klein’s account actually bolsters the claims of Husserl’s phenomenology in that it shows it has the methodological resources to uncover the true nature of formalization.

§ 3. Husserl on Sedimentation and the Crisis of the Sciences

In order to further frame Klein’s accomplishments, let us take a closer look at Husserl’s account of sedimentation and the problems sedimentation presents for the sciences. As noted, Klein does not agree with everything in Husserl’s account, but it will nevertheless be worthwhile to consider Husserl’s own words on the matter, first in “The Origin of Geometry” and then in

The Crisis of European Sciences.

16 Ibid., 528.
In the “The Origin of Geometry,” Husserl takes geometry as an exemplar of all scientific knowledge involving ideal meanings and shows how its historical development inherently involves a process of progressively building up layers of meaning. This process begins in the use of language to express and share meanings amongst contemporary practitioners of the science and then is exacerbated by its historical development through time. As these layers build up, deeper ones remain covertly operative even when lost to sight due to being covered over by further developments of meaning-layers. Insofar as these hidden layers of meaning are not completely lost, however, they can potentially be recovered or reactivated by bringing back to light the original insights that first gave rise to them.

A sedimented meaning is thus one whose signification is only passively given in that its self-evidence is not actively awakened in the mind, although it may retain the possibility of being reawakened. Thinking with such meanings involves merely passive understanding of the grounds of their meaning. In Husserl’s words, “Passivity in general is the realm of things that are bound together and melt into one another associatively, where all meaning that arises is put together passively.” Moreover, this kind of thinking is unavoidable not only in the sciences but also in ordinary life, in which we tend to think by mere associations, whereby we lose sight of the originally intuitive self-evidence of meanings.

In scientific thinking such passive association can only be avoided by ensuring that the original grounds of a meaning can be reactivated and maintained. It is therefore incumbent upon individual scientists to maintain the accessibility of these grounds if the science is to keep its truly meaningful character. This is made all the more difficult and important, however, by the

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fact that scientific meanings build upon each other and thus are always taken over through the handing-down of tradition. As Husserl says,

> it is of the essence of the results of each stage not only that their ideal ontic meaning in fact comes later [than that of earlier results] but that, since meaning is grounded upon meaning, the earlier meaning gives something of its validity to the later one, indeed becomes part of it to a certain extent. Thus no building block within the mental structure is self-sufficient; and none, then, can be immediately reactivated [by itself].\(^\text{18}\)

The historical structure of a science thus poses an even greater challenge to the scientist in that reactivating the inherited meaning-structures of a science requires reawakening the whole chain of meanings and their evidence. Trying to reactive all the sedimented meanings of a science would thus hinder its actual progress, if for no other reason than the amount of time and energy it would require of its practitioners.

Is such a reactivation of the full chain of meaning even possible? Husserl proposes as a “fundamental law” that if the premises of a science can be reactivated so can the consequences that follow from them self-evidently.\(^\text{19}\) Thus the original genuineness of the self-evidence at the starting point of a science is carried through its development, although this is inevitably lost sight of for the very reason that the whole historical-logical chain of development cannot be kept in mind at one time. Rather, the sciences unavoidably involve an ever-advancing formation of meaning that contains documented within it the sediments of its past. Yet once this sedimentation begins it only begets more sedimentation in that meanings whose self-evidence is not awakened can only produce further such meanings. The ability to reactive these hidden sedimented meanings is therefore of great importance, as that is the only way to genuinely know what a science truly means.

\(^{18}\) Ibid., 363.

\(^{19}\) Ibid., 365.
Yet this is where the deepest problems begin for Husserl, as he believes the whole modern age lacks “the developed capacity” for reactivating the original self-evidence of a science like geometry, and thus cannot know whether its sedimented idealities have a genuine meaning that can be “cashed in.”20 The sciences, propelled forward by their practical success and utility, are taught and function by passing on ready-made concepts without emphasizing the grounds of the meaning of those concepts. As a result, the actual capacity for reactivating meaning from primal sources—a capacity that involves making a meaning directly intuitable—has not been handed down with the concepts that are in need of being grounded. Thus the whole system of a science appearing as a tradition is merely a claim of truth-meaning until its primary meanings are reactivated, as it cannot be taken for granted that what is traditionally given is necessarily reactivatable, and this is true of all sciences.

It is this situation that explains the need for “epistemological grounding” of the sciences.21 For Husserl, that need can only be met by connecting epistemological and historical investigations. Every act of making the grounds of a meaning evident is an act of “historical disclosure” in that it requires uncovering the original self-evidence that has been covered over by layers of sedimentation.22 For history, according to Husserl, is nothing other than “the vital movement of the coexistence and the interweaving of original formations and sedimentations of meaning.”23 This points to the need for an even larger task, namely, the disclosure of “the historical a priori as the universal source of all conceivable problems of understanding,”24 but

20 Ibid., 366.
21 Ibid., 368.
22 Ibid., 370.
23 Ibid., 371.
24 Ibid., 373. Husserl’s discussion of this topic covers pages 371-74.
Husserl’s considerations in “The Origin of Geometry” end with only a brief discussion of this issue.

Let us turn now to the “crisis” of the sciences with which Husserl is concerned. What is this crisis and how is it connected to sedimentation? We have already begun to see the seeds of the problem in the foregoing, but if we take up The Crisis of European Sciences we can get a clearer picture.

Husserl begins that work by situating the crisis of the sciences in the larger context of a crisis in modern European philosophy that is itself the cause of a crisis of European humanity as such. That the sciences are in crisis is apparent from the fact that they cannot speak to the meaning of human existence, of which there is much need:

The exclusiveness with which the total world-view of modern man, in the second half of the nineteenth century, let itself be determined by the positive sciences and be blinded by the “prosperity” they produced, meant an indifferent turning-away from the questions which are decisive for a genuine humanity. . . . [Science] excludes in principle precisely the questions which man, given over in our unhappy times to the most portentous upheavals, finds the most burning: questions of the meaning or meaninglessness of the whole of this human existence.25

This state of the sciences is a result of the positivistic turn their development took, whereby they abstracted from everything subjective, focusing instead solely on objective physical reality. Moreover, this development occurred because of the relationship between modern science and modern philosophy.

On Husserl’s account, modern science gets it first impulses in the Renaissance founding of modern philosophy, wherein the individual sciences were understood to be “dependant

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branches” of the one, universal philosophy. Beginning with Descartes, “this new philosophy seeks nothing less than to encompass, in the unity of a theoretical system, all meaningful questions in a rigorous scientific manner, with an apodictically intelligible methodology, in an unending but rationally ordered progress of inquiry.” This, however, eventually led to the positivistic understanding of the sciences as it is only “mere facts” that are susceptible to this approach. Accordingly, the positive sciences became more and more focused on such facts, while metaphysical questions, questions about the status of reason, were left aside because they were not susceptible to such an approach. Yet insofar as the sciences have their source—both historically and as truth-claims—in the ideal of a universal philosophy of which they are merely a part, they are only as grounded as is that philosophy. Thus, if the status of reason itself remains unclarified, the sciences are left unmoored.

This is precisely the situation according to Husserl. As modern philosophy failed to keep pace with the advancement of the positive sciences, it became more and more skeptical about the possibility of metaphysics. This, in turn, led to the rise of skepticism, which insists that the only validity to be found is in the factual experience of the world, as well as to a corresponding degradation of the belief in reason. This is the way in which modern philosophy itself is in crisis; it is no longer in touch with its founding principles and has nothing to replace them with. This, in turn, leads to the crisis of European humanity as such in that man’s loss of faith in reason entails the loss of faith in himself. This, then, is the way in which the crisis of the sciences, of philosophy, and of modern man are all deeply connected.

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26 Ibid., 8.
27 Ibid., 8-9.
28 Ibid., 9.
In response to this crisis Husserl calls for a clearer self-understanding of our current situation by means of critical historical reflection aimed at recovering the founding impulses of modern philosophy and modern science, as well as at how that starting point led to the current crisis. He begins that process in Part II of the *Crisis*, and there he makes explicit the connection between the crisis of the sciences and sedimentation.

To begin, Husserl claims, we must understand that the modern goal of a universal philosophy involves a transformation of an ancient idea. What sets the modern form of this goal apart from the ancient one is the idea of a universal, infinite task for the sciences, which is dependent on the invention of formal mathematics (occurring in the 15th and 16th century). While the ancients did have idealized mathematics, they did not have formal mathematics and thus could not conceive of the corresponding idea of an infinite yet systematic approach to mastering knowledge. It is only with the rise of this idea in modern mathematics that such a goal can come to dominate the sciences in general.

From [the invention of the new mathematics], thanks to the boldness and originality peculiar to the new humanity, the great ideal is soon anticipated of a science which, in this new sense, is rational and all-inclusive, or rather the idea that the infinite totality of what is in general is intrinsically a rational all-encompassing unity that can be mastered, without anything left over, by a corresponding universal science. . . . Its rationalism soon overtakes natural science and creates for it the completely new idea of *mathematical natural science*—Galilean sciences, as it was rightly called for a long time. As soon as the latter begins to move toward successful realization, the idea of philosophy in general (as the science of the universe, of all that is) is transformed.29

The invention of modern mathematics and the rise of modern philosophy work together therefore to give modern science its form, and the development of the latter shapes in turn the development of these, its two original influences.

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29 Ibid., 22-23.
In a lengthy section on Galileo’s mathematization of nature, Husserl goes on to show how nature itself came to be idealized by the modern impulse toward mathematical science. This process is heavily dependent upon Galileo’s appropriation and use of both traditional geometry and contemporary developments in arithmetic. In examining this aspect of Galileo’s thought, Husserl says, we must be very attentive to what he took for granted in the meaning of these sciences, for such meanings were sedimented in his use of them in that they were hidden from his view but implicitly included in that use.

Most pertinent in Galileo’s acquisition and use of geometry is the conflation of the space of geometry and the space of our actual experience. This conflation depends upon two things: first, on the idealization generative of geometrical objects, whereby the sensible realm is superseded in favor of the mathematical one, and second, on the application of these geometrical idealities to bodies of the sensible world, whereby a new kind of objective knowledge about the world is obtained. This back-and-forth movement allows for “a new kind of inductive prediction,” namely, calculation of unknown events from known ones. “Thus ideal geometry, estranged from the world, becomes ‘applied’ geometry and thus becomes in a certain respect a general method for knowing the real.” This development raises the possibility of a universal idealized causality that makes everything in nature precisely determined and thus mathematically knowable. In using the traditional geometry as a part of his natural science, Galileo takes all this

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30 It is important to note that Husserl admits to “simplifying and idealizing the matter” in hanging his entire analysis of the mathematization of nature on Galileo (Ibid., 57). As Klein says: “In analyzing the foundations of Galileo’s physics, Husserl does not intend to give a detailed historical account. Galileo’s name is, in this connection, somewhat of a collective noun, covering a vast and complex historical situation”; Klein, “Phenomenology and the History of Science,” 79. For a discussion of Klein’s disagreement with Husserl’s account of Galileo, see Hopkins, Origin, 56-61. Husserl’s account is discussed here, not with full endorsement, but because it is essential to his formulation of the notion of sedimentation and the crisis it causes.
31 Husserl, Crisis, 33.
32 Ibid.
for granted, thereby opening the door for the substitution of mathematical idealities for the sensibly-experienced real world.

On the numerical side the most important unclarified presupposition of meaning is found in Galileo’s use of formulae as pre-established correlations among mathematical idealities. This is the “decisive accomplishment” that allows for the “determined, systematically ordered predications” that go beyond our actual experience. In this way, the application of such formulae to the world is essential to the method of modern mathematical science. In their fully developed form, however, these formulae are dependant upon the algebraic formalization that occurs with modern algebra’s “arithmetization of geometry,” a process that “leads almost automatically . . . to the emptying of its meaning” in that the new mathematical objects created thereby have merely “a displaced ‘symbolic’ meaning.”

When this way of operating with empty formulae becomes methodological, it leads to a universal formalization in the sense of a purely formal theory of analysis. As this way of operating is developed, it becomes a mere calculating technique, a set of operations according to “rules of the game,” which loses sight of what originally gave those operations truth and meaning. This results in what Husserl calls a “technization” of method, whereby one can calculate simply by following technical rules without understanding the genuine meaning of that process or the results it generates. Insofar as modern science becomes concerned with idealized formal objects, it too is susceptible to this technization of method and has indeed fallen prey to it, leaving it with merely superficial, empty meanings.

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33 Ibid., 43.
34 Ibid., 44-45. This process is analyzed in great detail by Klein and will be discussed further below.
35 Ibid., 46.
This technization and the accompanying emptying of meaning occurred because of the substitution of mathematical idealities for the objects of our actual experience whereby nature itself became idealized and substituted for the prescientifically intuited “life-world.” Accordingly, modern mathematical science represents the life-world by dressing it up in a “garb of ideas” and symbols, making it objective, and taking this garb to be true. This substitution leads to the mistaken understanding that nature itself is mathematical, when in reality this is a conflation of method and “true being.” Moreover, this conflation renders unintelligible the true meaning of that method and the theories produced from it because the actual self-evidence on which this development rests is lost in the sedimentation that gives rise to that very use of method. Thus to truly uncover the lost meanings, one must go past the idealities of modern science, back to the true source of meaning in the life-world, which on Husserl’s view remains unchanged by the technical developments.

Given that the true meaning of the method has been lost in its being handed down as a tradition, the achievements of natural science can only be truly meaningful if one inquires back into the original meanings at work in the historical establishment of their inherited meanings. Without this, the sciences are no longer in touch with the reality they think they are revealing.

We have now seen the way in which the crisis of the sciences is directly tied to the loss of meaning that comes from the sedimentation inherently at work in their historical development. Husserl goes on to discuss the role of dualism and Descartes in the development of natural

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36 Ibid., 49.
37 Ibid., 51.
38 Ibid.
science, but for present purposes we have now adequately considered Husserl’s own understanding of the issues at hand.\(^{39}\)

§ 4. The Importance of Formalization and the Task of Desedimentation

It is the result of Klein and Husserl’s investigations that one cannot address the questions of the meaning, history and importance of modern mathematics without considering the role that formalization played in creating and shaping it. Moreover, it is also a result that formalization itself cannot be understood without an investigation into its historical origins in the invention of modern algebra, as the two turn out to be concomitants. Thus, without an historical understanding of formalization and its dependence on algebraic symbols, one cannot grasp the nature of modern mathematics as such.

Furthermore, insofar as the conceptual structure of modern mathematics is not understood, the conceptual structure of modern mathematical science cannot be understood either because the latter presupposes the entirety of the former along with its problems. This lack of conceptual clarity thereby extends to the whole modern approach to understanding ourselves and the world, as modern science is one of the most dominant influences on modernity as such. Modernity rests on modern science, which rests on modern mathematics, which rests on the invention of formalization, which rests on the use of algebraic symbols, which (we will show) turns out to rest on the peculiarly Cartesian understanding of the relationship between

\(^{39}\) Husserl discusses dualism in §§ 10-11 of the *Crisis*, while Descartes is discussed in §§ 16-21. Our own discussion of dualism and Descartes is contained in the following chapter.
mind and world. Without a grasp of the connection and interdependence of the various parts of this chain, one cannot fully understand any of its individual members.

Hence the need for desedimentation—the historical investigation of scientific concepts aimed at uncovering the layers of meaning upon which those concepts are predicated—as an essential aspect of investigating the meaning of modern science and modernity at large; for that is the only way to reveal how these various layers of meaning have been built up over the modern period. Thus, without a clear account of the issues surrounding formalization any attempt to wrestle with the nature of modern mathematics and the sciences which use it is ultimately doomed to come up short.

This highlights the importance and necessity of the kind of historical-philosophical studies that Husserl called for and Klein performed. These two thinkers opened up a new line of investigation aimed at clarifying the status of modern mathematical thought by investigating its origins and developments in order to see the way they shape our current everyday understanding. Husserl laid out the groundwork for this approach by providing the philosophic rationale for desedimentation, while Klein actually began the inquiry by desedimenting formalization itself. In light of their accomplishments, one can now follow in this line, building upon their work, by extending their insights to other areas and periods of the development of modernity in its mathematical and scientific forms.

While it is the goal of the present study to do exactly that, we must first discuss Klein’s accomplishments in greater detail. The discussion up to this point will have prepared us to more fully appreciate Klein’s work, and the following will leave us better prepared to confront Descartes’s concept of space.
§ 5. Klein on Ancient Versus Modern Conceptuality

We begin our discussion of Klein’s major accomplishments with a look at his most overarching conclusions regarding the differences in ancient versus modern thought, which he frames in terms of the “conceptuality” (*Begrifflichkeit*)\(^40\) of the two sides. While the precise meaning of this term is a bit difficult to pin down, the general sense is that of the manner in which concepts are formed and used, or more simply, how concepts are given their meaning. This can be seen from the range of meanings Klein connects with this term, which he generally applies in two different ways: pretechnically, to the everyday manner in which concepts are used, and technically, to the distinct conceptual structures of Greek mathematics and modern (algebraic) mathematical physics. Regarding the former, Klein connects conceptuality to the “mode of thinking” (*Denkweise*) or “mode of conceiving” (*Auffassungsweise*) involved in “our approach to an understanding of the world.”\(^41\) In connection with the latter, Klein speaks of “the most general conceptual presuppositions of Greek arithmetic and logistic,”\(^42\) the “concept formation in our science,”\(^43\) the “conceptual means Greek, in distinction from modern, mathematics employs,”\(^44\) the “specific conceptual character” of “formal ‘algebraic’ symbolism,”\(^45\) and “the particular character of the concepts” used in modern mathematical

\(^{40}\) In *Greek Mathematical Thought and the Origin of Algebra* the German *Begrifflichkeit* is translated variously as “intentionality,” “conceptualization,” and “concept formation,” but the translator herself, Eva Brann, has since endorsed the use of “conceptuality” (See Brann’s “Preface” to Hopkins, *Origin*, xxiv). For this reason, along with the fact that it is the more standard translation of this word, I use “conceptuality” to capture the meaning of Klein’s *Begrifflichkeit*. For a further discussion of the translation of this key term, see Hopkins, *Origin*, 80 n. 7.

\(^{41}\) Klein, *GMTOA*, 118.

\(^{42}\) Ibid., 117.

\(^{43}\) Ibid., 120.

\(^{44}\) Ibid., 127.

\(^{45}\) Ibid., 128.
physics.\footnote{Klein, “World of Physics,” 7. David Lacterman, the translator of this lecture, follows Brann in translating \textit{Begrifflichkeit} as “intentionality.”} Klein also glosses conceptuality as “the way in which . . . concepts intend what is meant by them whenever they are employed,” by which he means to highlight that concepts can have different kinds of objects to which they are variously related.\footnote{Ibid., 6. This issue of the relationship between a concept and its object is an important aspect of conceptuality for Klein and it will be taken up in detail in the following section. As the above list of quotations makes clear, however, the meaning of this term is not limited solely to the mode of intending an object. It should be noted that the comment in brackets on page 118 of \textit{Greek Mathematical Thought and the Origin of Algebra} that reads “By intentionality \textit{[Begrifflichkeit]} is meant the mode in which our thought, and also our words, signify or intend their objects” is inserted by the translator, as are the two accompanying references (as well as the immediately preceding gloss of \textit{Begrifflichkeit} as “the manner of dealing with concepts”).}

These two aspects of conceptuality, the everyday manner of thinking or understanding and the conceptual structure of a particular science, turn out to be closely intertwined in that they are mutually influential. This relationship comes to light in Klein’s comparison of ancient science and its accompanying conceptuality with modern science and its own conceptuality. According to him, the “reinterpretation of the ancient body of doctrine, which brings with it a characteristic transformation of all ancient concepts, lies at the foundations not only of all concept formation in our science, but also of our ordinary intentionality \textit{[Begrifflichkeit]}, which is derived from the former.”\footnote{Klein, \textit{GMTOA}, 120.}

Klein’s analysis of the ancient and modern concepts of number reveals there to be a clear conceptual transformation that underlies the differences between ancient and modern mathematics, which he believes to be emblematic of substantial differences in the way the ancients and the moderns go about understanding the world in general. This difference has not been adequately recognized according to Klein, and it is thus one of his major goals in recovering the ancient concept of number to bring out this larger difference in ancient and
modern conceptuality. In this way, his attempt at revealing the differences in the ancient and modern concepts of number shows the close connection between the particular conceptuality of a science and the general conceptuality of the age that shapes it.

The failure to recognize the conceptual differences in ancient and modern mathematics has, on Klein’s account, hindered a proper evaluation of both sides; for he believes that one cannot properly understand a science without understanding the conceptuality that informs it. Insofar as ancient conceptuality is not appreciated, an understanding of ancient mathematics on its own terms is impossible. As Klein shows, modern attempts to understand ancient mathematics presuppose the conceptuality of modernity, and this makes them unable to see the ancient side clearly. Hence Klein’s attempt to recover what he thinks the standard interpretations of the history of mathematics fail to understand.

This failure to understand ancient mathematics on its own terms leads, in turn, to problems in understanding modern mathematics, as the conceptuality of the latter can only be fully understood in opposition to the ancient mode it replaces. This is in large part because modern conceptuality is taken for granted, assumed to be entirely adequate to its task without any real effort to understand it. To make the distinction between ancient and modern mathematics clear, it is therefore necessary to uncover the roots of modern conceptuality by investigating its origin and seeing how it arose in tandem with modern mathematics. Only by putting this origin in contrast with the ancient mode can the two sides be seen clearly. So long as that contrast is lacking, modern conceptuality itself will remain taken for granted by our whole way of thinking, thereby limiting our ability to understand our own conceptual presuppositions.
A proper evaluation of both sides is therefore impossible without recovering ancient conceptuality and putting it in contrast with modern conceptuality. Klein’s analysis of ancient versus modern mathematics does just that and in doing so shows that there is indeed a substantial difference in the whole approach each side takes to understanding the world.

We can begin to see this difference in the alternative ways the concepts of ancient and modern science are grounded and given their meaning. Klein shows that ancient science is rooted in the pre-scientific, natural understanding of the world and its concepts are formed accordingly. Even while trying to rise above such foundations in the purely contemplative attainment of knowledge, ancient science is continually informed by these grounds and preserves them intact in that it maintains a basis in the direct experience of the world. Modern science, on the other hand, stems from an impulse to reject the traditional science it inherited. Its original roots thus lie in the opposition to the already established science it aims to overthrow, and the meaning of its concepts comes from such opposition. Modern science is therefore natural in the sense that it aims to return to nature, which it believes has become obscured, rather than being rooted in natural experience and the understanding such experience gives rise to. Moreover, this difference in grounding is connected with another significant point of contrast, namely, that the primary motivation of modern science lies in solving problems by the practical application of knowledge rather than in disconnected theoretical understanding.49

With this we begin to see the major difference that marks the two sides of our contrast: the ancient mode is grounded in the world and continually connected to it, while the modern

49 For Klein’s discussion of the various, interconnected issues mentioned in this paragraph, see ibid., 118-22.
mode is no longer immediately in touch with the world but rather functions at an operational remove from it. This leaves modern science only indirectly related to the natural world.

This indirect connection highlights another important aspect of modern science that sets it apart from ancient science: its emphasis on generalization. Klein connects this aspect of the modern approach to knowledge with its character as an art or craft (in the sense of téchne).

Given the goal of artful manipulation of the world, a greater generality of knowledge is preferable in modern science as such knowledge is more widely applicable and, accordingly, more powerful. Modern science thus consciously aims for a kind of generality that Klein shows to be foreign to the concepts of ancient science, namely, a generality of the object of investigation rather than a generality of method. In line with this, ancient and modern conceptuality are importantly different in that modern conceptuality is equipped to deal with such general objects, while the ancients were not and in fact could not have been given that such general objects cannot exist according to the nature of their conceptuality. Modern conceptuality, on the other hand, is developed precisely for the sake of handling the new general objects that are concomitant with the invention of modern mathematics. As Klein says, the “intimate connection between the mode of ‘generalization’ of the ‘new’ science and its character as an ‘art’” is “exemplified in the symbolic formalism and calculational techniques of modern mathematics.” As will be discussed in detail later, the generality at work in this “symbolic formalism” is first and foremost one of indeterminateness. In fact, it is precisely this

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50 This distinction will be fleshed out more fully in the ensuing sections. For now it suffices to see that ancient science employed only determinate and therefore non-general objects, while modern science employs indeterminate and therefore general ones; both, however, employ methods that are general in that they allow for the investigation of multiple objects at the same time (e.g., a proposition about all triangles).

51 Klein, GMTOA, 122.
indeterminacy of conceptual objects that modern conceptuality makes possible in a way unthinkable from the ancient point of view.

The particularly modern form of these indeterminate concepts is ultimately the result of a different conception of the relationship between the mind and the world, which also sets modern conceptuality apart from ancient conceptuality in a fundamental way. Here we get to the heart of the difference between the two ways of understanding the world, as it is the alternative accounts of the mind-world relationship that are responsible for the direct versus indirect knowledge of the world.

At the core of ancient conceptuality is the belief that the mind is immediately in touch with the world and receives it directly. As Klein says of the ancient mode: “though there is a clear distinction between the mind and the world, there is no separation between them, . . . rather mind is very emphatically the receiving of the world and nothing but that.”

This understanding of the mind-world relationship accounts for the two important features of ancient conceptuality considered above, namely, its rootedness in the natural experience of the world and the determinacy of its concepts. Insofar as the ancients understood the mind to be directly in touch with the beings of the world, their whole mode of trying to understand that world is premised on the mind’s ability to take up those beings as they are.

In substantial contrast with this, the account that underlies modern conceptuality takes the mind to be separate from the world while being indirectly in touch with it in a way that makes it

52 Jacob Klein, “Modern Rationalism,” in Lectures and Essays, ed. Robert B. Williamson and Elliott Zuckerman (Annapolis: St. John’s College Press, 1985), 58. This is, of course, a rather large claim and Klein does not argue for it at length. It is well beyond the scope of the present investigation to attempt such an argument, but Aristotle’s account of the mind and its activity in On the Soul III.4-8 is a good starting point for confirmation of this thesis.
still knowable. The best-known version of this understanding of the mind-world relationship is the dualism of Descartes, about which Klein says the following:

This dichotomy [between thought and the external world] involves a profound distrust of the reality of the world. The mere fact that we question the possibility of receiving the outside world and the manner in which it may be received, that is, the very existence of the theory of knowledge, indicates the deep cleavage between mind and the outside world. The fact of supreme importance is that we consider our mind as a mind shut up within its own cell, that we consider our soul as a soul isolated and without any possible contact with the outside world. Hence the paradox that the mind which is taken to be all sufficient for understanding the world is preconceived as being entirely dissociated and alienated from the world.\footnote{Ibid., 57-58.}

According to Klein, the dichotomy at work in Descartes holds true of all modern conceptuality; Descartes was simply the one who most actively proclaimed it. While it is the case that the doctrine of substance dualism was discarded by the later tradition, Klein shows that Descartes’s understanding of the mind-world relationship is built into the nature of modern mathematics and thus also modern mathematical physics, and through the dominance of the latter has come to be characteristic of modern conceptuality as a whole.\footnote{This too is a rather large claim in need of a great deal of argumentation. Klein lays out a good portion of that argument in explaining how the Cartesian account of abstraction in the \textit{Rules for the Direction of the Mind} is essential to the formalization that underlies modern mathematics. This will be discussed and reviewed in detail in the next chapter, but it should be mentioned here that the separation of mind from world is not simply equivalent to substance dualism; as we will see, it is the former without the latter that is at work in the \textit{Rules}.} Moreover, it is precisely this operational remove of the mind from the world that accounts for the modern mode of concept formation in that the meaning of all modern concepts is shaped by the process through which the separation between mind and world is overcome. As we will see later, it is precisely that process that gives rise to the indeterminacy of conceptual object that is characteristic of modern conceptuality.

This, then, is the true difference between ancient and modern conceptuality, whether the mind apprehends the beings of the world directly or indirectly. Ancient conceptuality is based...
on the understanding that the mind is immediately in touch with and receptive of the world, which means that its concepts are based in natural experience. Modern conceptuality is based on the understanding that the mind is removed from the world and thus its concepts get their meaning in the way they are applied to the world from that remove. In this way we can begin to see how the different understandings of the mind-world relationship serve as the foundations that shape the way each side goes about using the mind and its concepts to understand the world.

Klein points to a number of important consequences that result from the conceptual transformation at the roots of modernity. As the new conceptuality of modern mathematics comes to shape the whole of modern science, it plays a large role in the overall process whereby the ancient mode of understanding the world is overthrown by the modern one.\(^5\) One chief result of this change is that mathematical physics now plays one of the most important roles that ancient philosophy used to play, namely, that of “the fundamental ontological science.”\(^5\) Moreover, as a result of the invention of the new indeterminate conceptual object mentioned above, one of the most significant ontological questions of ancient philosophy—the question of the mode of being of mathematical objects—is “obviated at one stroke.”\(^5\) Modern science, on the other hand, rests on the symbolic formalism of modern mathematics, “whose ontological presuppositions are left unclarified.”\(^5\) This leads modern science to operate in “a new conceptual dimension” that is “a medium beyond reflection,” wherein all the concepts of the new science get

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\(^5\) Hopkins succinctly describes the upshot of Klein’s thesis about the transformation in conceptuality as follows: “prior to this transformation what things are was not understood to be conceptual and linguistic, while now it is” (Origin, 4).

\(^5\) Klein, GMTOA, 184-85. See also “World of Physics,” 26-27.

\(^5\) Klein, GMTOA, 122.

\(^5\) Ibid., 184.
their meaning only in relation to each other. Even more broadly, the ancient conception of the world as an ordered whole, a *cosmos* with a *taxis*, is replaced by that of a world structured merely as “a ‘lawfully’ ordered course of ‘events,’” an understanding heavily dependent on the invention of algebraic formulae.

These are the major issues Klein raises that point to the need for greater conceptual clarity in modern science arising from its dependence on modern mathematics and its obscure presuppositions. While the broader ramifications of the rise of modern conceptuality are well beyond the scope of the present investigation, they are worth highlighting, both to see what is at stake in Klein’s analysis and because the present investigation is ultimately connected with a number of them.

§ 6. Klein on First and Second Intentions

In bringing modern conceptuality to light in contrast with ancient conceptuality, Klein employs the scholastic distinction between first and second intentional concepts, or more simply, first and second intentions (*intentiones primae* and *intentiones secundae*). As we saw at the beginning of the previous section, in his lecture “The World of Physics and the ‘Natural’ World” Klein directly connects conceptuality with “the way in which . . . concepts *intend* what is meant by them whenever they are employed.”

This question of how and to what kind of object a concept is related is precisely the issue at stake in the distinction between first and second intentions.

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59 Ibid., 121.
60 Ibid., 185. For a slightly longer discussion of this issue, see “World of Physics,” 32-34.
Klein gives a straightforward definition of these concepts later in that lecture. First intentions, he says, are “concepts which refer immediately to individual objects,” while second intentions are “concepts which refer directly to other concepts, to intentiones primae, and only indirectly to objects.” In *Greek Mathematical Thought and the Origin of Algebra*, Klein gives a similar definition, glossing a second intention as “a concept which itself directly intends another concept and not a being.” The intended object of a second intention thus exists in the mind, while the intended object of a first intention is “a ‘being’ which is directly apprehensible.”

Klein identifies a further aspect of the distinction between the two kinds of intentional objects when he indicates that the object of a first intention has “actual objectivity” while the object of a second intention is merely a “general object.” He also speaks of the object of a second intention as an “abstract being” (*ens abstractum*) or a “being of reason” (*ens rationis*), which owes its existence “to the operation of the intellect alone.”

It is in explicating this last point that Klein refers to a scholastic schoolbook, the *Summa Philosophiae* of Eustachius a Sancto Paulo, as his source for the terminology of first and second intentions. Klein first quotes Eustachius’ definition according to “established usage” wherein a second intention is understood as “an *ens rationis* ‘which is conceived as belonging to a thing known by virtue of its being known, and which cannot exist except as something present in the

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62 Ibid., 17-18.
64 Ibid., 175.
65 Ibid., 192.
66 Ibid., 207.
67 Klein quotes from a selection of Eustachius’ book found in Etienne Gilson, *Index Scholastico-Cartesien* (Paris: Alcan, 1912), which indexes subject matters found in the works of Descartes and gives related passages from Scholastic writings. It is worth highlighting that Klein’s scholastic reference for the terminology of first and second intentions comes from a work on the conceptual background of Descartes.
intellect, since it is conceived [not originally but] secondarily and by a reflexive operation of the mind." He then gives Eustachius’ more etymological definition in an endnote: “a second intention, if you look at the meaning of the name, is that very operation of the mind by which it secondarily intends a thing already conceived before, insofar as it has been conceived.”

Eustachius thus emphasizes the mind’s act of conceiving in his discussion of second intentions, whereas Klein tends to focus more on the nature of the object at which the mind is directed in its conceiving. Yet Klein himself seems to indicate that Eustachius’ two definitions show that the term “second intention” may refer either to the reflexive activity of the mind or to its object in such activity. For the sake of clarity one should try to maintain this important distinction between a second intention and its object.

We have thus seen that a first intention is a concept that intends, refers or applies to a being or an individual object, while a second intention is a concept that has as its object another concept, which as such exists only in the mind. While this is the heart of the distinction between the two kinds of concepts, what is more important for Klein is the difference between the specificity of the object of a first intention versus the generality of the object of a second intention. Indeed, whenever Klein brings up second intentional concepts he connects them with the issue of the generality of their intended objects.

To understand this better, we turn to Klein’s brief discussion of first and second intentions in his lecture “Modern Rationalism,” which starts with the now familiar distinction

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68 Klein, GMTOA, 207.
69 Ibid., 306 n. 324.
70 Hopkins has a nice reformulation of this: “First intentions concern the existence and quiddity of an object, its being in its own right; second intentions concern an object insofar as it has being in being known, in apprehension” (Origin, 508).
between concepts “which apply to objects and those which apply to concepts themselves.”

What is particularly helpful about this passage is that Klein goes on to give generic examples of both. As examples of a concept being applied to an object, he considers the statements “This is a dog” and “This is red.” In these sentences, “this” is an actual object being named in speech, while “dog” and “red” are concepts that are being applied to that object. “Dog” and “red” are thus the first intentional concepts, while the “this” to which they are applied would be the object of a first intention. As examples of concepts being applied to other concepts, Klein takes the statements “Red is an attribute” and “Idealism is a theory.” Here “attribute” and “theory” are concepts that are applied to other concepts, namely, “red” and “idealism;” the former are second intentional concepts, while the latter are the objects of second intentions as well as first intentional concepts in themselves.

Klein goes on to stress that the distinction between first and second intentions is not based on degrees of abstraction:

Every concept is, as such, abstract. That is, the concept is drawn, is abstracted, from individual objects, is general in itself, and has its own reality only in the mind. Abstraction, an Aristotelian term, means a process of our thought by which, for instance, the concept “dog” is drawn from individual dogs as something common to them all, or the concept of a mathematical triangle is drawn from objects of triangular shape.

Here we get a clear picture of how Klein understands concepts. On his account, all concepts are generalities that are drawn from individuals and exist only in the mind. As such, however, concepts can refer either to things outside the mind, which are always determinate or specific beings, or to other things within the mind, namely, other concepts which are always general. With this, we can see the significant point of contrast regarding generality between first and

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71 Klein, “Modern Rationalism,” 60.
72 Ibid.
second intentions. While both kinds of concepts are general in themselves, their objects differ on this score. First intentional concepts are themselves general but their objects are determinate, while second intentional concepts are both general in themselves and applied to general objects. There is, therefore, an important distinction as regards generality, but it lies solely in that to which the concepts are applied: determinate objects or general concepts.

This points to another important aspect of the contrast that is worth highlighting: first intentional concepts refer directly to individual objects, while second intentions refer only indirectly to such objects insofar as they refer first to another concept. First intentions are thus not only more determinate, they are also more immediate in the sense that second intentions exist at a further remove in that another concept stands between them and a determinate object.

From all this we can see that the difference in kind of intentional object has important ramifications for the nature of first versus second intentions. This is not to say that either kind of intentional concept is problematic, but rather that each has its own particular characteristics. It is quite important, however, that the differences between the two be clearly understood, otherwise they could be used confusedly. As we will see in the next section, the modern concept of number rests precisely on such a use.

§ 7. Klein on the Modern Symbolic Concept of Number and the Abstraction that Gives Rise to It

Klein’s desedimentation of the modern concept of number is intimately connected with his detailed analysis of the early-modern mathematicians who invented it. Without getting too caught up in that analysis, however, we can still understand the structure of this concept by
employing the distinction between first and second intentions, as it is precisely that distinction
that allows Klein to uncover the nature of the modern symbolic concept of number in its contrast
with the ancient one.

It is the result of Klein’s careful analysis of the ancient concept of number, the Greek
*arithmos* concept, that it is always first intentional. Within Greek conceptuality a number is
always “a definite number of definite things.”73 Thus, when the mind is dealing with such a
number, it is always directed at an assemblage of determinate individual objects.74 In fact,
according to Klein ancient mathematics in general, including geometry, is first intentional in the
sense that it always deals with determinate objects,75 and he even goes so far as to suggest that
the same is true of all ancient science.76

The modern concept of number, in contrast, is not first intentional but rather symbolic, by
which Klein means something very particular involving the conflation of first and second
intentional objects. The “symbolic understanding” of an intended object occurs when a second
intentional object, namely, a general concept, is taken as a first intentional object, which is
determinate, resulting in “the identification of the mode of being of the *object* with the mode of
being of the *concept* related to the object.”77 Such a conflation is dependent on the use of a
drawn sign, which is itself a determinate first intentional object, to represent a second intentional
object, which is general. “The heart of the symbolic procedure,” according to Klein, is thus
constituted by two things: “identify[ing] the object represented with the means of its

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73 Klein, *GMTOA*, 46.
74 It is an important part of Klein’s analysis that these individuals can be either physical objects of the material world
or pure mathematical monads in the intelligible realm. The ontological status of the latter is the subject of great
debate between Plato and Aristotle, as Klein penetratingly explores.
75 See Klein, “World of Physics,” 12-17.
76 See Ibid., 17 and “Modern Rationalism,” 60.
77 Klein, *GMTOA*, 192.
representation” and “replac[ing] the real determinateness of an object with a possibility of making it determinate.” By this process the indeterminacy (or merely possible determinacy) of a second intentional object is made determinate by conflating the general object with the determinate sign used to represent it. This results in a symbolic concept. Insofar as such a concept is dependent on the conflation of a first and second intentional object, it is also dependent on the conflation of the specificity versus generality of such objects. Here we begin to see the peculiar indeterminacy of the conceptual object that marks modern concept-formation.

What, then, does the symbolic concept of number look like? In showing that the invention of the modern concept of number is concomitant with the invention of algebra, Klein shows that this concept is fundamentally dependent on the conflation of the general concept of a number with the algebraic sign representing it. This can be seen in his analysis of François Viète’s “analytic art,” wherein he shows that Viète’s use of letter signs is fundamentally different from that found in ancient arithmetic. Whereas the ancient arithmetical letter sign simply stands for a specific arithmos and thus always immediately intends a determinate amount of monads, “[Viète’s] letter sign intends directly the general character of being a number which belongs to every possible number, that is to say, it intends ‘number in general’ immediately, but the things or units which are at hand in each number only mediately.” The “truly decisive turn,” however, which makes Viète’s letter sign function so differently from its ancient counterpart, is that the “general character of number” that the letter sign stands for “is accorded a certain

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78 Ibid., 123.
79 Klein’s argument for this point is found in his detailed analysis of the arithmetical procedure of Diophantus in Section 10 of Greek Mathematical Thought and the Origin of Algebra.
80 Klein, GMTOA, 174.
independence which permits it to be the subject of ‘calculational’ operations.”  

The algebraic letter sign thus stands for a general concept but is taken as “a ‘being’ which is directly apprehensible” and thus manipulable in the course of calculation.

In discussing Descartes, Klein gives an even more exact account of the conceptual character of algebraic symbols while also diagnosing the peculiar kind of abstraction that gives rise to this new concept of number, which he terms “symbolic abstraction.” Here we begin to see the intimate connection between the Cartesian understanding of mind and the modern mode of concept-formation. Descartes’s account of this abstraction will be considered in detail in the next chapter, so for now we will simply outline the structure of symbolic abstraction in order to get a clearer picture of the nature of the modern symbolic concept of number.

It must be mentioned first, however, that Klein places heavy emphasis on the newness of this kind of abstraction, saying that Descartes “postulates—with an explicitness perhaps novel in the history of sciences—a new mode of ‘abstraction’ and a new possibility of ‘understanding.’” Klein even goes so far as to suggest that Descartes is the only thinker who has ever tried to account for the particular kind of abstraction that underlies the symbolic concept of number: “Descartes and, as far as we can see, only Descartes, struggles to fix the exact meaning of such an ‘abstraction.’”

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81 Ibid.
82 Ibid., 175.
83 Klein’s German is symbolische Abstraktion. This is rendered as “symbol-generating abstraction” in the English translation, which according to Brann—the translator—Klein endorsed (see Brann’s “Preface” to Hopkins, Origin, xxvii n. 3). Hopkins, however, argues for retaining the term “symbolic abstraction” (Origin, 306-07 n. 165). Following Husserl, Hopkins also refers to this form of abstraction as “formalizing abstraction” (Origin, 524-31).
84 Klein, GMTOA, 200.
85 Ibid., 295-96 n. 314. Hopkins echoes this claim: “Indeed, Descartes’ account . . . remains the first and to this day only philosophical attempt to fix the exact meaning of the abstraction that yields the new, algebraic number concepts that are employed . . by the new ‘symbolic’ mathematics” (Origin, 501-02). It is worth noting, however, that in
The heart of Descartes’s account of this new kind of abstraction is the peculiar relationship between the imagination, which he understands to be a bodily organ, and the “pure intellect,” which is understood to be completely distinct from the body. From this it is clear that we are here dealing with the Cartesian mind-body separation. As entirely distinct from the body, the pure intellect can have no involvement with images; rather, as “pure,” it is concerned only with simple intelligibles. The imagination, however, being itself bodily, can supply bodily images to the pure intellect, which is precisely what it does in the process of symbolic abstraction. In this way, the imagination plays an essential role in bridging the gap between the mind and the world. As Klein puts it: “The ‘pure’ intellect, in order to be at all able to come into ‘contact’ with the objects of the corporeal world . . .—that is to say, in order to come in contact with the ‘world’ in general—needs the mediation of a special faculty, namely, precisely that of the imagination.”

Here we are thus concerned with Descartes’s attempt to overcome the previously mentioned paradox, namely, “that the mind, which is supposed to be sufficient to understand the world, is preconceived as a mind alienated form this same world.”

So how does the relationship between intellect and imagination work to conflate first and second intentions in order to give rise to the new concept of number? Insofar as the imagination is bodily, it always deals with determinate beings, and thus when it comes to number it deals building upon Klein’s accomplishments and giving his own articulation of formalizing abstraction, Hopkins himself gives an alternative account that is grounded in the phenomenological tradition (Origin, 526-28). The existence of an alternative account shows that Descartes’s account is not the only one possible, and thus that it is not necessary to explain symbolic abstraction. This point will be returned to and made more specific in the next chapter after we have discussed Descartes’s account of symbolic abstraction at length (see Footnote 46 of Chapter 2), but to foreshadow the important implication: Hopkins’s account shares some similarities with Descartes’s but does not depend upon all of its details, which serves as an initial indication that we do not have to accept all the details of Descartes’s account as true in order to explain modern mathematics.

86 Klein, GMTOA, 202-3. Klein here describes the relation between body and soul as “the insoluble problem of Cartesian doctrine.”

87 Klein, “Modern Rationalism,” 63.
with a determinate multitude of units, which is to say its relationship to number is first intentional. The pure intellect, on the other hand, is completely distinct from the body and deals only with ideas that are similarly “pure”; thus, as regards number, the pure intellect can have only a second intentional, general concept of number. These differing imaginative and intellectual dealings with number are connected, however, in that the general concept of number is obtained by the intellect from the imaginative representation of determinate multitudes.

When the imagination presents a number of units to the pure intellect, the latter can separate out the general concept of “mere multitude” or “multitudinousness-as-such” (sola multitude), which Klein describes as “the ‘naked’ indeterminate manyness to which simply nothing ‘true,’ nothing truly in ‘being,’ and hence no ‘true idea’ of a being corresponds.” In considering this concept, which is clearly second intentional, the pure intellect is directed at its own act of knowing and thus is immediately concerned with an ens rationis and only indirectly with actual objects. In this way, the very act of knowing becomes the focus of the mind’s cognitive regard, whereby the act of knowing itself is apprehended “as a ‘something,’ namely, as one and therefore as an ‘ens,’ a ‘being.’” This process gives rise to the “the ‘concept’ of the number as such.”

Insofar as this general concept of number is second intentional and therefore indeterminate, it has no mathematical properties in itself, yet both Viète and Descartes deal with such general concepts mathematically in the new mathematical procedures they invent. In order for these concepts to enter into mathematical operations they need to be represented by

88 Klein, GMTOA, 201.
89 Ibid, 208. Hopkins points out that “Properly speaking, the formalized intentional object does not present an object at all but rather the apprehension of the apprehension of a multitude of objects” (Origin, 527).
90 Klein, GMTOA, 208.
something determinate, otherwise the pure intellect could not deal with them in a determinate fashion. The imagination here plays a second role in relation to the pure intellect, namely, providing it with such a determinate representation in the form of a sign. In Viète, these signs take the guise of letters and Arabic numerals, while in Descartes they are given an additional form of lines and figures. When such abstract, general concepts are given such concrete, determinate representations, however, second intentional objects are being taken up in the mode of a first intention.

This might not necessarily be problematic in itself if the sign were merely meant to call to mind the general concept for which it stands. When the determinate representation of the indeterminate concept of number is treated as a mathematical object, however, as it is when Viète and Descartes make it manipulable in mathematical operations, the second intentional concept is conflated with the first intentional object that represents it. This conflation of a general and indeterminate concept with a particular and determinate object is the heart of what Klein understands a symbol (in his technical sense) to be: “When now—and this is of crucial importance—the ens rationis as a ‘second intention’ is grasped with the aid of the imagination in such a way that the intellect can, in turn, take it up as an object in the mode of a ‘first intention,’ we are dealing with a symbol.”91

This symbol-generating process, resulting in a determinate representation being conflated with the indeterminate concept it represents, is what Klein calls symbolic abstraction. While his account of this abstraction is based in Descartes alone, he believes the Cartesian account underlies the modern concept of number as such. This is the case because the symbolic concept-

91 Ibid.
formation that underlies that concept depends upon the conflation of a general concept (which cannot exist in the world but only in the mind) with the determinate object that represents it (which as determinate is a material object of the world). Symbolic concept-formation thus inherently involves a separation of intelligible content and material representation, as well as a very particular collapsing of that separation. Even without the full-blown substance dualism of Descartes, symbolic concepts therefore entail a gap between what is in the mind and how that is taken up in the world. It is in this way, then, that symbolic abstraction shows an essential connection between the Cartesian understanding of mind and the modern mode of concept-formation.

As we have already seen, according to Klein the historical development of the modern concept of number is a crucial element in the rise of modern conceptuality as a whole; so too, therefore, is the symbolic abstraction that underlies it. In the next section, we will consider the important role Descartes plays in extending that development beyond the realm of the mathematical.

§ 8. The Centrality of Descartes in Klein’s Analysis

We have already caught a few glimpses of Descartes’s importance in Klein’s account of modern conceptuality. Not only is Descartes influential in the creation of the modern symbolic concept of number, he also provides an epistemological account for its possibility. Moreover, Descartes is not content to leave this symbolic concept in the realm of mathematics; through his attempts at founding a new mathematical science, he is responsible for connecting this otherwise mathematical concept with the corporeal world. Thus, as mathematician, philosopher and
scientist, Descartes plays a central role in connecting these various realms in a way that is hugely influential on the shape of modernity as a whole.

Having already rehearsed Descartes’s account of symbolic abstraction, a word must be said about his contribution to the development of the modern concept of number, as he is also responsible for a major step in that development. The details of this contribution will be discussed in Chapter 3, so for now we simply highlight Klein’s comments on the matter. According to Klein, Descartes’s “original achievement” is extending to the realm of geometry the same reinterpretation of the traditional concept of number that occurred in the arithmetical realm at the hands of Viète and others. 92 Yet, Klein continues: “The truth is that Descartes does not, as is often thoughtlessly said, identify ‘arithmetic’ and ‘geometry’—rather he identifies ‘algebra’ understood as symbolic logistic with geometry interpreted by him for the first time as a symbolic science.” 93 Descartes thus broadens the symbolic understanding, originally created in the realm of arithmetic, to cover the whole of the mathematical domain, and it is this that makes him one of the most important pioneers of modern mathematics.

Even more important, however, is the fact that this extension of the new symbolic concept of number to the realm of geometry is precisely what allows Descartes to connect the mathematical and physical realms in his attempt to found a new mathematical physics. In doing so, Descartes puts forward the radically new idea that symbolic mathematics and physical science are concerned with the same thing. As Klein puts it, Descartes’s “great idea” is to identify the general object of algebra, understood as the universal science (mathesis universalis), with extension itself, understood to be the very substance of corporeal beings, and it is only by

92 Klein, GMTOA, 206.
93 Ibid.
this identification that “symbolic mathematics gain[s] that fundamental position in the system of knowledge which it has never since lost.”

Here we begin to see the full extent of the influence Descartes has on the shape of modernity. Klein spells this out in even more detail:

[A]bove all—and it is this which gives [Descartes] his tremendous role in the history of the origin of modern science—he was the first to assign to “algebra,” to this “ars magna,” a fundamental place in the system of knowledge in general. From now on the fundamental ontological science of the ancients is replaced by a symbolic discipline whose ontological presuppositions are left unclarified. This science, which aims from the first at a comprehension of the totality of the world, slowly broadens into the system of modern mathematical physics. Within this discipline the things in this world are no longer understood as countable beings, nor the world itself as a taxis determined by the order of numbers; it is rather the structure of the world which is grasped by means of a symbolic calculus and understood as a “lawfully” ordered course of “events.” The very nature of man’s understanding of the world is henceforth governed by the symbolic “number” concept, a concept which determines the modern idea of science in general.

So not only does Descartes help create the modern concept of number, he identifies that new mathematical object with the physical substance of the world as a philosophical justification of mathematical physics, whereby he gives modern symbolic mathematics a fundamental position in coming to knowledge, which eventually comes to dominate modernity at large. It is in this way, then, that Descartes plays a major role in extending the symbolic concept-formation that began in the mathematical realm to the physical and beyond.

It is the goal of the current inquiry to contribute to the understanding of this role by highlighting the importance Descartes’s concept of space has in this process. This concept holds a central position in his thought in that it is essential for his tying together of the mathematical and physical realms. Moreover, we aim to show that this concept is itself symbolically-formed

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94 Ibid., 197-98.
95 Ibid., 184-85.
in that its creation is dependent upon symbolic abstraction. Klein provides us with a glimpse of this in the few brief remarks he makes on the subject of space, which we consider in the next section as they serve as the inspiration for the present study.

§ 9. Klein on the Modern Concept of Space

The concept of space first appears in its new symbolic form in the thought of Descartes. It is one of the primary concepts, both temporally and in importance, to arise as a further development of the new symbolic mode of concept-formation. In considering this concept, we are thus dealing with the key example that allows us to see the transition whereby modern conceptuality transcends its roots in the mathematical realm and moves into the physical.

Klein makes only a few scant remarks about the Cartesian concept of space and its importance, but they set the stage for our own investigation. In Greek Mathematical Thought and the Origin of Algebra, Klein connects the concept of space directly with extension’s role of uniting the mathematical and physical domains in the thought of Descartes:

Extension has . . . a twofold character for Descartes: It is “symbolic”—as the object of “general algebra,” and it is “real”—as the “substance” of the corporeal world. More exactly, in Descartes’ thinking, the dignity of representing the substantial “being” of the corporeal world accrues to extension precisely by reason of its symbolic objectivity within the framework of the mathesis universalis. Only at this point has the conceptual basis of “classical” physics, which has since been called “Euclidean space,” been created. This is the foundation on which Newton will raise the structure of his mathematical science of nature.

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96 There are of course precursors to the Cartesian concept of space, but, as we will argue, this concept is dependent upon symbolic abstraction for its mathematical-physical form and thus is radically modern. It follows that none of these precursors anticipates the Cartesian conception in the decisive respect. For a good history of the concept of space, see Max Jammer, Concepts of Space: The History of Theories of Space in Physics, 3rd ed. (New York: Dover Publications, 1993).

97 Klein, GMTOA, 210-11.
Klein spells this out in a bit more detail in “The World of Physics and the ‘Natural World.’” There he says that the peculiar concept-formation that underlies modern algebra is essential to understanding that “Descartes’ concept of *extensio* identifies the extendedness of extension with extension itself.”

He continues:

> Our present-day concept of space can be traced directly back to this. Present-day Mathematics and Physics designate as “Euclidean Space” the domain of symbolic exhibition by means of line-segments, a domain which is defined by a coordinate system, a relational system, as we say nowadays. “Euclidean Space” is by no means the domain of the figures and structures studied by Euclid and the rest of Greek mathematicians. It is rather only the symbolic illustration of the *general character of the extendedness* of those structures. Once this symbolic domain is identified with corporeal extension itself, it enters into Newtonian physics as “absolute space.”

With these two quotations, we can begin to see that Descartes’s concept of space is a symbolic formation that serves as the conceptual grounding of mathematical physics.

While we will not be able to cash out the above quotations fully until the end of our investigation, let us unpack them a little here in anticipation of what is to come. Klein’s claim is that Descartes’s concept of extension encompasses and connects the symbolic object of modern mathematics and the nature of the physical world. Moreover, this concept is itself symbolic in that it conflates extendedness-as-such (“the extendedness of extension” or “the general character of extendedness”—i.e., the general property of being extended, which is the object of a second intention) with extension itself (i.e., that which is extended, which is the object of a first intention). This leads to the modern concept of space when this symbolic understanding of extension is taken as the domain in which all extension exists. This domain can be articulated

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99 Ibid.
100 Perhaps Klein’s most surprising claim about space is that the modern understanding of “Euclidean space” is actually this symbolically conceived domain of extension, which is mistakenly read back onto ancient geometry. To
by line-segments in the Cartesian coordinate grid, and insofar as mathematical and physical extension are the same, this articulated domain encompasses both the physical and mathematical such that it can serve as the conceptual framework within which mathematical physics occurs. This framework thus serves as the foundation upon which classical physics is built, as exemplified in Newton.101

In the following chapters this account of the modern concept of space will be spelled out and argued for in detail from within the works of Descartes. Our ultimate goal is to show that symbolic abstraction underlies the Cartesian concept of space and that any attempt to understand the nature of this concept must consider the role that symbolic-concept formation plays in its creation. The key to this will be looking, first, at this formation itself (along with its conceptual presuppositions) as laid out in Descartes’s *Rules for the Direction of the Mind*, and then at how the concept of space is employed in Descartes’s *Geometry* and connecting that with his discussion of space as equivalent to extension and the substance of the corporeal world in the *Principles of Philosophy*. In this way we will see how the symbolic concepts of Descartes’s mathematical thinking extend to his understanding of the material world in a way that is influential on the shape of modern science in general.

 unpack this claim, let alone argue for it, would require more time than can be given to the matter here. Suffice it to say that a careful study of Euclid with this question in mind would be necessary to truly address the issue. 101 Joseph Cosgrove rightly points out that “Klein adduces no specific evidence from ‘Newtonian science’ itself to back up the claim that the latter is somehow essentially dependent upon Cartesian geometry” and that such a claim therefore needs “to be cashed in historically”; Joseph Cosgrove, “Husserl, Jacob Klein and Symbolic Nature,” *Graduate Faculty Philosophy Journal* 29 (2008), 236. The desedimentation of the Cartesian concept of space, intended by the current investigation, would be a necessary precursor to any such attempt at “cashing in” its role in later, Newtonian physics.
§ 10. Klein’s Accomplishment and How to Carry it Forward

Before we turn to Descartes, let us review what we’ve now covered of Klein and his work. Against a Husserlian background, we saw that Klein began the project of desedimenting the conceptual structure of modern mathematics by uncovering the historical origins of formalization in the invention of symbolic algebra. This turned out to be dependent upon the Cartesian understanding of mind, as that understanding is built into the peculiar kind of abstraction that underlies this formalization, upon which depends the very possibility of the symbolic concept of number. Moreover, that concept is at the heart of modern mathematics, which is in turn at the heart of modern mathematical science, which is in turn at the heart of modern conceptuality as such. Thus, insofar as symbolic abstraction underlies the modern concept of number, its conceptual presuppositions are built into the whole modern approach to understanding the world. In uncovering all this, Klein thus began the task of providing the conceptual clarification of which modernity is in need.

While Klein clearly has his sights set on the entire chain that begins with formalization and runs up to modern conceptuality as such, his desedimentation is confined primarily to the mathematical realm. If one is persuaded that Klein is right about both the beginnings of this chain and the connection between its links, the task one is left with is to extend his work beyond the realm of mathematics to the mathematical sciences in an attempt to move his analysis closer to modern thought in its current form. This, of course, is a long and arduous process that must move step-by-step through the development of modern science and its concepts to see how they’ve become what they are today; for it is by tracing the historical development of the sciences that we reveal the layers of sedimentation upon which they have been built. Only in this
way can the present state of the sciences and their powerful influence on everyday understanding be clarified.

We turn now to our own attempt at furthering Klein’s project by extending his accomplishments beyond the mathematical realm. As was just said, the task Klein leaves us with is to further desediment various meanings and concepts of modern mathematical science in light of his analysis of the modern concept of number and the concept-formation that goes with it. It is the goal of the present study to do exactly that with the modern concept of space by uncovering its nature and origin in the thought of Descartes. While Klein has provided us with some indications of how to connect his account of the modern concept of number with Descartes’s concept of space, his claims about the matter are merely preliminary and he has not argued for them. To develop and justify them requires the detailed textual analysis needed to ground them in the works of Descartes, which is precisely what is accomplished in the following chapters.

To begin the process of uncovering the nature of Descartes’s concept of space, we first must explicate the conceptual presuppositions of Descartes’s mathematics. Such an explication is essential background to an analysis of Descartes’s concept of space because that concept turns out to be an extension of Descartes’s new mathematics. This explication will be accomplished in Chapter 2 by means of an analysis of the account of mathematical cognition contained in the *Rules for the Direction of the Mind*, along with the understanding of the mind-world relationship upon which that cognition is predicated.\(^\text{102}\) Throughout the analysis of Chapter 2 we will be

\(^{102}\) In this way, Chapter 2 is aimed at an analysis of the presuppositions of Descartes’s concept of space, rather than at an analysis of that concept itself. In fact, space is not much of an issue in the *Rules*, and, accordingly, there is no
unpacking and grounding claims that Klein made but did not fully support. In Chapter 3, we then go beyond Klein’s analyses by investigating the *Geometry*, first, to show that the account of mathematical cognition contained in the *Rules* is at work in Descartes’s mature mathematics, and, second, to show that the understanding of space in the *Geometry* is itself a further product of that mathematical cognition. Chapter 4 then takes up an examination of the physical understanding of space in the *Principles of Philosophy* and shows that it is related to the mathematical understanding of space in the *Geometry* in such a way that allows the physical and mathematical realms to be united as a grounds for mathematical physics. Chapter 5 then brings our investigation to a close by highlighting its major conclusions.

direct discussion of space in Chapter 2. Let the reader be forewarned that this concept is therefore not taken up directly until Chapter 3.
CHAPTER 2

Mathematical Cognition and the Relationship Between Mind and World in Descartes’s *Rules for the Direction of the Mind*

§ 1. Introduction to the Present Analysis of the *Rules for the Direction of the Mind*

The goal of the present chapter is to uncover the understanding of the mind-world relationship at work in the *Rules for the Direction of the Mind*¹ and show how it is built into the account of mathematical cognition contained therein. It is important to note at the outset that this work contains the only attempt in Descartes’s corpus to give an epistemological grounding for his use of symbolic concepts, as this highlights the necessity of focusing on this early text to gain an understanding of the conceptual presuppositions of Cartesian mathematics. While there are a number of important themes and ideas connected with the issues of symbolic mathematics that are also taken up in Descartes’s later works, we cannot simply impute the understandings found in those works back into the *Rules*.² For this reason, the focus of the following analysis is confined exclusively to this work.

The *Rules* is an early, unfinished, and posthumously published treatise, however, and this has given rise to many controversies regarding various aspects of the text, beginning with the

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¹ This is the traditional translation of the title of this work. It should be noted, however, that “mind” here translates “ingenium,” which is better rendered as “natural intelligence.” For a discussion of the use and meaning of “ingenium” in the early writings of Descartes, see Dennis L. Sepper, *Descartes’s Imagination: Proportion, Images, and the Activity of Thinking* (Berkley: University of California Press, 1996), 84-96.

² The later development of such themes and ideas is thus not immediately at issue here. There is a wealth of literature on how various aspects of Descartes’s thought develop from the *Rules* into what is found in his later writings, but such considerations will only be discussed when they are immediately germane to the matter at hand, as the detailing of any of those historical developments is well beyond the scope of the present investigation.
details of its composition. There are, accordingly, a number of scholarly debates regarding the nature and cohesion of the work. Rather than try to settle such debates, the following will proceed simply by examining the text itself, which will ultimately be seen to yield enough cohesion of its parts, as well as consistency of its ideas, to allow a clear picture of the pertinent issues to emerge.

In order to explicate those issues, the following analysis begins with a section that frames the investigation of the mind-world relationship in the *Rules* within the larger context of the question of Cartesian dualism (§ 2). This is then followed by the main body of the analysis, which consists of a number of sections spent working through those parts of the text that are pertinent to the various issues at hand, namely, the preliminary indications of the mind-world relationship in Rules 1-3 (§ 3), the enumeration of faculties in Rule 8 (§ 4), the full account of the knowing faculties in Rule 12 (§ 5), the establishment of the mathematical character of the remainder of the *Rules* in Rules 12 and 13 (§ 6), and the account of mathematical cognition in Rule 14 (§ 7). The analysis then culminates in a section discussing how the mind-world relationship in the *Rules* range from 1619-1629. Jean-Paul Weber has argued that different sections of the text were written at substantially different times throughout that period, thereby raising questions about the relation of its parts and the coherence of the whole; see Jean-Paul Weber, *La Constitution du Texte des Regulae* (Paris: Société D’édition D’enseignement Supérieur, 1964). While Weber’s thesis seems to have gained general acceptance it is not without its detractors, including notable scholars such as Jean-Luc Marion and Dennis Sepper; see Jean-Luc Marion, *Sur l’ontologie grise de Descartes: Science Cartesien et Savoir Aristotélicien dans les Regulae* (Paris: Vrin, 1975), 16-17; Dennis L. Sepper, “Descartes and the Eclipse of the Imagination,” *Journal of the History of Philosophy* 27 (1989): 387 n. 14; Sepper, *Descartes’s Imagination*, 37-38 n. 3 & 146-52. For an overview of the major contours of the debate, see John A. Schuster, *Descartes-Agonistes: Physico-mathematics, Method & Corpuscular-mechanism 1618-33* (Dordrecht: Springer, 2013), 227 & 235-38; and Chikara Sasaki, *Descartes’s Mathematical Thought* (Dordrecht: Kluwer Academic Publisher, 2003), 190-94.

This presumably puts the present study of the *Rules* in the camp of those interpreters who believe it to be something more than a collection of different historical strata, although my intention is simply to let the text speak for itself, thereby allowing it to determine whether or not it is cohesive. As far as I can tell, nothing in the following analysis is undermined by a developmental understanding of the text. For an argument for the thematic unity of the *Rules*, see Pamela Kraus, “Whole Method’: The Thematic Unity of Descartes’ *Regulae,*” *The Modern Schoolman* 63 (1986): 83-109. For an account that argues for the cohesion of different historical parts of the *Rules*, see Schuster, *Descartes-Agonistes*, 328-330.
relationship found throughout the *Rules* is built into Descartes’s account of the symbolic concept of number (§ 8). The chapter then concludes by showing how the conceptual presuppositions built into Descartes’s account of mathematical cognition make their way into the mathematical practices sketched out in Rules 15-22 (§ 9).\

§ 2. The Mind-World Relationship in the *Rules*

While it is generally agreed that Descartes has a dualistic understanding of the relationship between mind and body, there is much debate as to the nature and status of that dualism. It is the contention of the following analysis that while Descartes’s conception of mind is inherently dualistic, it is not in the *Rules* at least metaphysically or substantially so. The dualism at work there, it will be shown, is rather just that “operational remove” of the mind from the world that was discussed in the previous chapter.

The basic tenet of substantial dualism is that the mind and the body are separately existing substances. Non-substantial dualism, by way of contrast, understands the two to be distinct in some other, metaphysically neutral way. Both are enough, however, to establish a “separation” between the mind and the body and, by extension, between the mind and the world. The difference lies simply in the nature of that separation. For this reason, the mind-world relationship can be understood as the general category within which the various forms of dualism

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5 In investigating all these issues and showing their interrelations, the following analysis is intended to spell out and fully justify Klein’s interpretation of the *Rules*, which exists in a rather condensed form that leaves his arguments somewhat truncated. It is worth noting that when Klein’s work on the *Rules* is acknowledged it tends to receive high praise; see, for example, Hiram Caton, *The Origin of Subjectivity: An Essay on Descartes* (New Haven: Yale University Press, 1973), 52; Schuster, *Descartes-Agonistes*, 325 n. 66; Sasaki, *Descartes’s Mathematical Thought*, 357 & 434.
are found, all of which take the mind to be separate from the world in some way. Accordingly, it is safe to say that Descartes has a generally dualistic understanding of the mind-world relationship, while the specific understanding of that dualism needs to be parsed out in the individual context of any particular Cartesian work, which is precisely what the following aims to do for the *Rules*.

The relationship between the non-metaphysical mind-world separation found in the *Rules* and the separation of full-blown substantial dualism to be found, presumably, in the latter writings of Descartes (most famously in the *Meditations on First Philosophy*) is a complicated question involving the historical development of Descartes’s thought over the course of his lifetime. While such a topic is beyond the scope of the present investigation, it is nevertheless clear that substantial dualism entails an operational remove of the mind from the world, but not vice versa. It is safe, therefore, to take up a consideration of non-substantial dualism without getting caught up in the issues of substantial dualism, while also being confident that the questions of the former will persist in the latter. This too should make it clear that the mind-world separation is the more pervasive aspect of Descartes’s thought.

The goal here, therefore, is not to enter into the scholarly debate on the nature of Descartes’s dualism, although the following is intended to make a contribution to the literature on that subject by highlighting the need to discuss non-substantial dualism. Although that need is largely missed by most scholars, there are a few rare exceptions on which the present analysis relies. Most notable among these are Richard Kennington and Pamela Kraus, who are the

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I prefer to speak of the mind-world relationship rather than the mind-body relationship because the primary question at hand, at least in the *Rules*, is how the mind has access to the world outside it, as opposed to the question taken up in Descartes’s later writings of how the mind knows first the body as another part of the self and only after that how it knows bodies other than the self.
biggest influences other than Klein on the reading of the *Rules* presented below. For this reason, we will briefly review some of their relevant conclusions regarding the status of Cartesian dualism as a way to frame our own analysis, which is taken up in what immediately follows.

In “The ‘Teaching of Nature’ in Descartes’s Soul Doctrine,” Kennington reaches the significant conclusion that “not the dualism of mind and body but that of mechanism and purposiveness (or of science and human experience) is the basic Cartesian dualism.” While this is not the dualism that will ultimately be identified in the *Rules*, in the course of arguing for this Kennington shows that there is an important distinction in the *Meditations* between Descartes’s separation of mind from body and his substantial dualism. This is evident, he argues, from the fact that in Meditation 2’s establishment of the *res cogitans*, Descartes “avoids any metaphysical determination as regards ‘substance.’” Instead, the mind of the thinking thing is “separated or distinguished only ‘methodologically’ . . . and in no sense ‘metaphysically,’” such that the *res cogitans* itself “is metaphysically neutral as regards mode of being, or substantiality.” The merely methodological separation of mind and body found in Meditation 2, which Kennington also calls “the standpoint of scientific intellect,” is thus markedly distinct from full-blown substantial dualism, which only appears later in the middle of Meditation 6. The establishment of this distinction between substantial dualism and Descartes’s separation of mind and body

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8 Ibid., 177.

9 Ibid., 179.

10 Ibid., 172.

11 Kennington’s analysis of Meditation 6 has important implication for the understanding of substantial dualism in the thought of Descartes in general (see ibid., 180-86), but given that substantial dualism itself is not a matter of concern to the present investigation, this aspect of Kennington’s reading will be left aside.
provides preliminary justification for leaving out the former in an analysis of the latter, although the ultimate justification for doing so comes from the *Rules* itself (as our own analysis will show).

Continuing in the line of interpretation initiated by Kennington, Pamela Kraus has identified a form of non-substantial dualism that is found in both the *Meditations* and the *Rules*. In an essay aimed at showing that the account of mind found in the *Meditations* is composed of two distinct components, Kraus identifies one of those components as a doctrine of cognitive faculties that is found in both Meditations 2 and 6 as well as in Rule 12. According to this doctrine the mind is capable of functioning both in conjunction with body and independently from body, although those functions can only be performed separately, for which reason Kraus speaks of this doctrine as containing a “functional dualism.” Moreover, Kraus shows that the doctrine of cognitive faculties is distinct from both Meditation 2’s understanding of the *res cogitans* and Meditation 6’s argument for the existence of mind as a separate substance. In this doctrine, Descartes thus presents an understanding of the mind as a single source of disparate functions, while its “metaphysical determination” is left up in the air. The functional dualism contained in this doctrine is therefore, again, markedly distinct from substantial dualism.

By identifying this non-substantial dualism in Rule 12, Kraus has prepared the way for our own analysis, which is aimed in part at showing that there is an understanding of the mind-

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13 Kraus, “*Mens Humana,***” 2, 11, 13 & 14.
14 Ibid., 15. Kraus makes similar claims regarding the metaphysical neutrality of this doctrine in her article “Whole Method,” which is focused exclusively on the *Rules*, as will be discussed in the following section.
world relationship pervading the *Rules* that entails a separation between the two. This mind-world separation will ultimately be shown to exist in a form of dualism that is merely operational and methodological but markedly not substantial. In this way, then, the following analysis follows the lead of Krauss and Kennington by identifying a non-metaphysical dualism that posits a separation of mind and body without determining the two as independently existing substances. Thus, insofar as the text of the *Rules* cannot support a metaphysical reading of the dualism it contains, it must be concluded that substantial dualism is absent from this early treatise and should not be read back into it from any later work.

§ 3. The Foundational Role of the Mind-World Relationship in Rules 1-3

The understanding of the mind as separated from the world is present from the very beginning of the *Rules*, appearing in the second sentence of Rule 1. There Descartes contrasts “the sciences, which consist entirely of knowledge possessed by the mind [*animi*]” with “the arts, which call for some practice and disposition of the body [*corporis*]” (359). Unlike the arts, Descartes says, the sciences should not be pursued individually nor should they be distinguished by differences of object. Rather, all the various sciences comprise human wisdom (*humana*

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References to the works of Descartes are given parenthetically in the body of the text by page numbers of the standard Adam-Tannery edition (typically abbreviated “AT”); *Oeuvres de Descartes*, 11 vols., ed. Charles Adam and Paul Tannery, 2nd ed. (Paris: Vrin, 1964-76). When a particular work is first introduced I will give the number of the Adam-Tannery volume that contains it and I will list any other editions and translations I have used. The translations given throughout will be my own modified versions of those I have consulted. The Latin text of the *Rules* is contained in Volume X of Adam-Tannery, to which all parenthetical citations in the main text of this chapter refer. In addition to this, I have also consulted the following critical texts and translations: *Regulae ad directionem ingenii. Texte critique établi par Giovanni Crapulli avec la version hollandaise du XVIIème siècle* (The Hague: Martinus Nijhoff, 1965); *The Philosophical Writings of Descartes*, vol. 1, trans. John Cottingham, Robert Stoothoff and Dugald Murdoch (New York: Cambridge University Press, 1985); *Regulae ad directionem ingenii – Rules for the direction of the natural intelligence: A bilingual edition of the Cartesian treatise on method*, ed. and trans. George Heffernan (Amsterdam–Atlanta, GA: Rodopi, 1998).
sapientia), which is one and the same regardless of how it is applied to different objects. All the individual sciences are therefore interconnected and interdependent, and they should thus be mastered as whole. This is the best way to cultivate a good mind (bona mens) and universal wisdom (universalis Sapientia), to which all the individual sciences contribute and for which everything else is to be valued. Based on this understanding, Descartes advocates that we aim at the general goal of cultivating the mind’s ability to make true and sound judgments, rather than at the particular goal of some specific knowledge. In a statement that serves as an articulation of the guiding principle of the Rules, he says “If one seriously wants to investigate the truth of things . . . one should simply think about increasing the natural light of reason [naturalis rationis lumine]” (361).

Rule 1 thus sets the stage for Descartes’s attempt to unify the sciences such that they can be pursued and mastered as a whole. He opens this work by contrasting the sciences with the arts because he thinks the traditional distinctions between the sciences stems from a false judgment about the similarity of the two. The Rules thus begins with a divide—one that the tradition has failed to make. On one side there is science, knowledge and the mind, and on the other there are the arts which come about by habituating the body. By dividing the knowing faculty from bodily learning at the very outset, Descartes prepares the understanding of mind that will become explicit later in the work, thereby setting it up to play a foundational role in his attempt to transform the sciences.

The mind-world separation is also present in the arguments of Rules 2 and 3. Rule 2 teaches that “[w]e should attend only to those objects for which the certain and indubitable cognition of our natural intelligence seems to suffice” (360). The only such objects Descartes is
aware of are those of arithmetic and geometry, as these two disciplines “alone are free from any
taint of falsity or uncertainty” (364). The mathematical sciences thus appear to be the only true
sciences of Descartes’ day. The reason arithmetic and geometry are “more certain” than any
other discipline is that “they alone are concerned with an object so pure and simple that they
presuppose nothing at all that experience might render uncertain, but consist entirely of
conclusions to be deduced rationally” (365). Consequently, Descartes says, “in seeking the right
path to the truth, one should be concerned with no object about which it is not possible to possess
a certainty equal to the demonstrations of arithmetic and geometry” (366).

Here Descartes prepares another major teaching of the *Rules*, namely, that mathematics
should serve as the model for all science. The mathematical disciplines hold this exemplary
position because of the purity and simplicity of their rational objects, which, in turn, means that
the bodily senses do not threaten to undermine the reliability of their knowledge. It is thus an
alignment with the mind as opposed to bodily experience that leads Descartes to emphasize the
place of mathematics in the pursuit of all knowledge. Here the contrast between mind and body
appears again in a foundational capacity as it turns out to be the guarantor of the certainty of
mathematical knowledge.

In the course of Rule 2, Descartes states that there are two ways of arriving at knowledge:
experience, which can be deceptive, and deduction, which cannot err (364-65). In Rule 3,
however, when naming “the actions of our intellect [intellectus] through which we are able to
arrive at a knowledge of things without any fear of being mistaken,” he excludes experience and
instead names only intuition and deduction (368). Experience cannot be on this list because the

16 As Heffernan notes in his translation of this passage, “experience” (experientia) can have a wide range of
meanings, but here it seems clearly connected with the bodily senses; see Heffernan. trans., *Regulae*, 73 n. 19.
knowledge it yields is unreliable. The replacement of experience with intuition in Rule 3 thus represents a further distancing of mind from body, as the experiential knowledge that comes from the body is thereby discarded in favor of intellectual activity.

This is born out further in the description of intuition that follows. In contrast to “the fluctuating testimony of the senses” and “the deceptive judgments of the imagination as it composes things badly,”\textsuperscript{17} intuition is said to be “the conceptual act \textit{conceptum} of a pure and attentive mind \textit{mentis} so easy and distinct that absolutely no doubt can remain about what we are understanding;” or, as Descartes also puts it, it is the “indubitable conceptual act of a pure and attentive mind, which springs from the light of reason alone” (368). Intuition is thus an act of the mind occurring independently from any involvement with the body.

Descartes goes on to say that of intuition and deduction, the former is simpler and more certain. In fact, intuition turns out to be the basis of deduction, as the latter consists in “that which is necessarily concluded from some other thing known with certainty” (369). Intuition is thus the starting point from which deduction begins, whereby it serves as the ultimate source for all certain knowledge. In this way, it provides the very foundation of that at which Descartes’s method is aimed. Insofar as this intuition is purely mental, it serves as yet another indication of the centrality of the mind’s independence from the world.

Pamela Kraus is helpful in further explicating the foundational role of the mind-world separation in these first three Rules. As she puts it, Rule 1 “establishes a fundamental doctrine

\textsuperscript{17} This locution clearly leaves open the possibility of the imagination composing things well in a way that leads to non-deceptive judgments. This possibility will later be seen to be central to Descartes’s account of mathematical cognition in Rule 14.
that the mind is naturally independent as a knower.”

She explains the propaedeutic role of this beginning by pointing out that this doctrine is intimately connected with Descartes’s denial that the mind is effected differently by different kinds of objects, which is in turn essential to his unification of the sciences. For that unification depends upon replacing the traditional “philosophical account” that understands the mind to be “naturally ordered to know different kinds of beings” with a new account of mind that “underwrites” and “guarantee[s]” the connection of the sciences by supplying “the correct relation of mind and world.” The establishment of this new account is therefore necessary for “[t]he success of the aim of the treatise” and it “occupies all but the last several rules of the Regulae.”

It begins, however, in Rule 1 with “the first and most basic thesis that the mind is the source of all truth.” Thus, on Kraus’s account, “standing both structurally and logically at the beginning of his project to join the sciences, is a non-metaphysical expression of Descartes’ view that the mind is independent of the body.”

Rules 2 and 3 then “redefine scientific truth in a way that is compatible with [Descartes’s] fundamental teaching about the mind.” They do this by spelling out the conception of truth that is implied by that teaching, namely, that, because the mind is the source of truth, it is only that which the mind can grasp directly and unproblematically that can be counted as true, whereby the domain of science is restricted to whatever the mind can grasp in this way. Together, Rules 2 and 3 thus establish that it is only objects appropriately suited to the

19 Ibid., 90.
20 Ibid.
21 Ibid.
22 Ibid., 91.
23 Ibid., 88.
mind and its activities that meet this criterion. Accordingly, Kraus concludes that “[t]he theoretical possibility of connecting the sciences ultimately rests upon the independence of mind as a source of truth.”

In Rules 1-3 Descartes thus lays out an interconnected web of ideas that governs the development of the rest of the Rules. Central to that web is the separation of mind from world that underlies the attempt to unify the sciences in the search for mathematically certain knowledge. This conception of mind is thus fundamental to Descartes’s project as a whole. While this conception is left largely implicit in these opening Rules, it is eventually brought to the fore and explicated directly in Rule 12. Between Rules 3 and 12, the mind-world separation makes only the occasional passing appearance. Rule 8, however, anticipates the account of Rule 12 by giving an outline of what will later be fleshed out more fully, and it will therefore be considered in the following section.

§ 4. The Enumeration of Faculties in Rule 8

Rule 8 advocates that we respect the limits of our mental powers by desisting from a line of inquiry as soon as we come upon something that “our intellect [intellectus] is unable to intuit” (392). Descartes gives an example to explicate this wherein he applies the rule itself to the question of “all the truths for the knowledge of which human reason [humana ratio] suffices”

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24 Ibid., 92.
25 See, for example, the complaint in Rule 4 that traditional mathematics “pertains more to the eyes and imagination than the intellect” (375), Rule 6’s distinction between intuiting simple natures “in experiences themselves or by some light situated within us” (383), and the preliminary division of simple natures in Rule 8 into the classes of “either spiritual or corporeal or belonging to both of these” (399).
26 Rules 4-7, which are not covered here because they do not shed any further light on the issues of primary importance to the present investigation, are concerned with the need for and nature of Descartes’s method, as well as the order, arrangement, and nature of the objects considered by that method.
(395). This examination, he says, is necessary once in a lifetime for the pursuit of a good mind (bonam mentem). It consists in an enumeration of our various knowing faculties, which—when spelled out fully in Rule 12—amounts to an account of the understanding of mind contained in the Rules. It should be highlighted at the outset, however, that Descartes here admits the epistemological account that supports his method is found by following the very approach laid out in the Rules, which is to say that he admits to generating his own supporting epistemological account from within the very approach he is advocating.

The examination of our intellectual powers is said to be “the finest example” of an application of this rule because “nothing can be known prior to intellect [intellectum], since the knowledge of all other things depends upon it” (395). The example is thus meant to delimit the boundaries of knowledge by delimiting the boundaries of the intellect itself. In an outline of this enumeration, Descartes lists three instruments of knowing (instrumenta cognoscendi) that correspond to three different modes of knowing (cognoscendi modis): the pure intellect (intellectus puri), the fantasy (phantasia) and the senses (sensus). Strictly speaking, he says, only the intellect contains truth and falsity, although the cause of these is often found in the other two.

This trifold division constitutes the first attempt in the Rules to enumerate the various capacities by which we know. While there is no statement made about the nature and relationship of these various capacities, it seems that “human reason” or “mind” is a broader category within which there are the three distinct instruments and modes of knowing. This initial understanding is complicated, however, by the presence of a different version of the same
account in the remainder of Rule 8, which is again only an outline. While these two versions are slightly different in their terminology and divisions, they will ultimately prove not to be contradictory in light of the fuller account given in Rule 12.

The second version begins with a repetition of the claims made at the beginning of the first version. Descartes says that in order to avoid uncertainty about what the mind (animus) can achieve, it is necessary once in life “carefully to have inquired what knowledge human reason [humana ratio] is capable of” (396-97). He then makes the self-supporting nature of the account all the more explicit by comparing his method to the mechanical arts which determine how their own instruments are to be made. The first thing to be done in light of this situation is to establish what the investigation of truth requires, and, Descartes says, nothing could be “more useful” to that endeavor than an inquiry “into what human cognition [humana cognitio] is and how far it extends” (397-98). This question is “the first of all to be examined by means of the rules already presented” because its investigation “contains the true instruments of knowing [instrumenta sciendi] and the entire method” (398).

This question, Descartes says, should be divided into two parts to be discussed separately: that which relates to us in our capacity for knowing and that which relates to the things that can be known. On the side of us as knower, “intellect [intellectum] alone is capable of knowledge,” but the intellect “can be aided or impeded by three other faculties [facultatibus], namely, the imagination [imaginatione], the senses [sensu], and the memory [memoria]” (398).

Regarding the things themselves as they are known, they “are to be considered only insofar as

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27 The presence of two versions of the same account (with slight variations) within the same Rule presumably reflects the various stages of composition. On the composition of Rule 8, see Weber, La Constitution, Chapter VI; and Schuster, Descartes-Agonistes, 311-14.
intellect reaches them,” and they are divided into “most simple” versus “complex or composite” natures, of which the simple natures are either “spiritual, or corporeal, or pertaining to both” (399). Upon completing this outline, Descartes then concludes by postponing a further clarification of both sides of the question until later.  

According to the second version of the enumeration, then, there is some general category of mind (referred to variously as animus, ratio, and mens) within which is the operation of the intellect in its act of knowing, which is in turn aided by the imagination, the senses and the memory. In contrast to the first version given in Rule 8, where the non-intellectual faculties were on par with the intellect (being co-listed with it as instruments of knowing), in the second version the other faculties are excluded from having knowledge; instead, their role is limited to assisting the intellect in its act of knowing, which is said to be the only true act of knowing. The addition of memory to the list of non-intellectual faculties also sets the two versions apart, as does the switch in terminology from phantasia to imaginatione. Yet there is still no indication as to the nature of these various faculties, although it is clear in both versions that the intellect is somehow set apart.

Rule 8 thus seems to be neutral on the question of the mind’s relationship to the body and the rest of the world. Nevertheless, Descartes there presents an outline of his account of our knowing ability, and it is already clear that the intellect holds a distinct position within that. This will become all the more clear in the spelled out version of this account in Rule 12, wherein Descartes’s understanding of the mind-world relationship becomes explicit.  

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28 Descartes seems to indicate that this will occur in the following Rule, but it is in fact delayed until Rule 12.
29 A discussion of Rules 9-11, which are primarily concerned with intuition and deduction, is omitted here because they, like Rules 4-7, are not immediately relevant to the matters at hand.
§ 5. The Account of Knowing Faculties in Rule 12

The discussion of knowledge in Rule 12 begins with a repetition of the prescription that we must separately consider what pertains to us as knowers and the things themselves that are known. Regarding the former, Descartes repeats the finding of the latter half of Rule 8 that there are only four faculties (facultates) that can be used for knowing—the intellect, imagination, senses and memory—among which the intellect alone is capable of perceiving truth, although it must be aided by the other three “lest we accidentally omit something which lies within our industry” (411). Concerning the things themselves, there are three matters to examine: what is spontaneously accessible, how one thing can be known from another, and what may be deduced from what.  

Descartes begins to address the question of our knowing abilities in Rule 12 by saying that he would like to explain what the human mind (mens hominis) is, what the body is, how the body is informed by the mind, what faculties there are in the composite whole that serve in the knowledge of things, and what those individual faculties do. There is not space for all that though, he says, and instead it will suffice to explain “as briefly as possible” what for present purposes “is the most useful manner of conceiving all that which is in us for the things to be known” (412). He even goes so far as to add the qualification that his readers should not believe his account is true unless they are inclined to do so, and if not, it can simply be taken as a supposition that makes things clearer without detracting from the truth of the matter.  

30 The part of Rule 12 concerned with this examination is less immediately relevant to the present investigation and thus will not be considered in detail here.
This qualification is rather surprising, and it is unclear how it should be taken. Does Descartes really mean that he does not think his account is true, or does he just want to protect against a negative reaction on the part of his reader? By offering this qualification, he seems to indicate that it is sufficient to take this account simply as the most useful way of understanding things in that it makes the matter clearer regardless of the account’s truthfulness, but one wonders whether Descartes can really endorse such a division between truth and clarity. The emphasis on utility, however, fits with Rule 8’s acknowledgement of the self-supporting nature of this account within the method advocated throughout the Rules. Here too Descartes seems more interested in the value of his account to his overall project of instituting a new approach for the sciences than in the truth or facticity of that account, whereby he gives a preliminary indication of neutrality regarding the ontological standing of what follows.

Descartes then commences his discussion of the various faculties, beginning with an account of sense perception. The senses are said to be parts of the body whose activity of sensing consists in being passively affected when a sense organ has the external shape of a sensed body impressed upon it. This is said to happen in exactly the same way that wax passively receives a seal-impression, for Descartes says explicitly that his appeal to wax and seal is not to be taken as a mere analogy. The sense organs are literally impressed with the shape of the body which they sense: “One must conceive the external shape [figuram]31 of the sentient body to be really [realiter] changed by the object in the same way that which is in the surface of

31 The Latin figura seems better translated as “shape” than “figure” in this context, but it will later have a decidedly mathematical character, in which case “figure” seems like the better rendering. Accordingly, I will move between both translations depending on the context, using “shape” when a more physical connotation is called for and “figure” when a more mathematical one is. This difficulty in translation reflects the fact that Descartes collapses the distinction between the physical “shape” of an external object and its internal representation by means of a mathematical “figure.”
the wax is changed by the seal” (412). Descartes also emphasizes that this is true of all the senses and their sensing, not merely the sense of touch. All sensation, then, consists in nothing but the physical impression of shapes on the various senses.

Conceiving of things in this way “helps greatly,” Descartes says, “because nothing more easily falls under the senses than shape,” and, moreover, “nothing false follows from this supposition more than from any other” (413). Taking color as his example and proposing we conceive of it in terms of shape alone, he suggests we simply abstract (abstrahamus) from everything else about it in order to focus solely on the fact that it has the nature of shape (figurae naturam). The same can then be done for everything else as well, “since the infinite multitude of shapes suffices for expressing all the differences of sensible things” (413). Descartes thus suggests that we conceive of everything that is sensible in terms of shape or figure. It should be underscored, however, that this is put forward somewhat tentatively and again with a greater emphasis placed on utility rather than truth.32

Continuing with his discussion of the bodily faculties, Descartes next raises the common sense. As the external senses receive an impressed shape, that shape is “conveyed to some other part of the body, which is called the common sense, in the very same instant and without the passing of any real entity from the one to the other” (414). This transmission is likened to the instantaneous motion conveyed from one end of a pen to another as it is used in writing. The common sense is thus itself a part of the body capable of receiving impressions, and it is the

32 The Rules does not explicitly contain Descartes’s later doctrine that the essence of body is extension (this doctrine will be discussed at length in Chapter 4’s consideration of the Principles of Philosophy); rather, in this early work it is only the internal reception and depiction of corporeal objects that is explicitly understood to be figural. Schuster rightly corrects Klein on this point; see Schuster, Descartes-Agonistes, 325 n. 66.
The instantaneous motion of the impressed shape from the sense organs that conveys impressions to it.

The common sense, in turn, “functions like a seal forming the same shapes or ideas . . . in the fantasy or imagination [phantasia vel imaginatione], just as in wax” (414). The imagination or fantasy is thus also a physical part of the body, likewise capable of receiving impressions. Descartes stresses that the imagination is large enough to take on many distinct shapes at the same time and that it can retain those shapes for some time, in which case it is called “memory”. The memory is thus not really a distinct faculty, but rather a mode of the imagination.

This completes the account of sense perception. External bodies impress their shape upon the sense organs, which convey that shape via seal-impressions to the common sense, which in turn conveys them to the imagination. Insofar as all that is received by the senses is the impressions of shape it is clear that all that exists in the imagination is also the impressions of shape, yet Descartes has spoken about the common sense as forming “shapes or ideas” in the imagination, and indeed he will continue to speak about ideas in the imagination throughout the rest of this account. These “ideas,” however, can be nothing other than impressed shapes and it is important that this be kept in mind.

Descartes then concludes his account of the bodily faculties with a discussion of “the motive force [vim motricem] or the nerves themselves” (414). The origin of these is said to be in the brain, where the fantasy is also located. The fantasy is responsible for moving the nerves in

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33 Descartes generally uses the terms phantasia and imaginatione interchangeably, but the former seems to have a more physiological connotation; see Heffernan, trans., Regulae, 145 n. 223.

34 This helps account for one of the differences between the two versions of the enumeration of faculties in Rule 8.
the same way that the common sense is moved by the external senses and the whole pen is moved by a part, namely, by instantaneous conveyance. The fantasy can thus cause many movements in the nerves “even though it may not have within it the express images of these, but certain others from which these movements can follow” (415). From this, Descartes says, we can understand how all movement that does not depend upon any cognition or reason happens in both animals and humans.

The point of this discussion of the motive force is to compliment the mechanistic account of sense perception with a mechanistic account of animate motion.\(^{35}\) Just as sense impressions can travel up the chain of bodily faculties by literally moving instantaneously from one to the other as a seal-impression, so can the impulse to motion travel from the imagination through the nerves into the various parts of an animate body. This emphasizes the corporeal and mechanistic nature of everything that has been laid out so far in this account. For Descartes has discussed nothing but the interactions of various parts of the body, and the only two descriptions of those interactions (those in terms of seal-impressions and moving pens) are physical and mechanical.

The first half of the mind-body separation has now been presented, and Descartes next jumps to the other side of the divide. It is important to note before moving on, however, that the imagination sits at the top of the chain of bodily faculties, as this sets up the important question of the interaction between the intellect and imagination, which will turn out to be the pivotal connection between the mind and the world.

Descartes begins his discussion of “the power [vim] through which we know things,” which he names the cognitive power (vis cognoscens), by presenting three features of how it is to

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be conceived: as purely spiritual (pure spiritualem), “no less distinct [distinctam] from the whole body than blood is from bone or hand from eye,” and one or singular (unicam—415). Despite the cognitive power’s singularity, however, it can perform a number of different functions: it can receive shapes from the common sense along with the fantasy, it can apply itself to those shapes after they are preserved in the memory, it can form new shapes in the imagination, and it can transfer such shapes to the motive force. In all this, the cognitive power is sometimes active and sometimes passive, which Descartes likens to sometimes resembling the seal and sometime resembling the wax, but he is explicit that here this is to be taken merely as an analogy since “something generally like this power is not found in corporeal things” (415).

While the cognitive power is one and the same (una et eadem) in itself, Descartes continues, in its various functions it is said to perform different operations and even be called different names:

if it applies itself [applicet se] with the imagination to the common sense, it is said to see, to touch, etc.; if to the imagination alone in so far as it is endowed with diverse shapes, it is said to remember; if to the imagination in order to form new shapes, it is said to imagine or conceive [concipere]; if, finally, it acts alone [sola agat], it is said to understand [intelligere] . . . . And the same power, then, according to these different functions [functiones], is called either pure intellect, or imagination, or memory, or sense; but it is properly named “natural intelligence” [ingenium] when, at one time, it forms new ideas in the fantasy or, at another, it attends to those ideas already made. And we consider this power suited to these different operations [operationibus]. (415-16)

So the cognitive power is one faculty that is able to act in a number of different ways, either by applying itself to the other faculties or by acting on its own. In the former case, the different applications of the cognitive power correspond to different functions or operations, whereby it is spoken about in different ways, even (perhaps confusedly) being given the same name as the various bodily faculties. When the cognitive power applies itself to the common sense in
conjunction with the imagination, it receives shapes from it; in performing this function it is said to sense and is even called sense. When it applies itself to the shapes that are preserved in the imagination, it is said to remember and is itself called memory. When it applies itself to the imagination so as to perform the operation of forming new shapes there, it is said to imagine or conceive and is itself called imagination. In either of these applications to the imagination, it is also properly called natural intelligence. When the cognitive power acts alone, however, it is said to understand and it is only in performing this function or operation that it is called the pure intellect.

It is only in its function as pure intellect, then, that the cognitive power acts separately from the body. In all its other functions or operations it works in conjunction with the bodily faculties by applying itself to them, in which cases the cognitive power works so closely with the bodily faculties that the distinction between the two seems difficult to maintain, as evidenced by the fact that the cognitive power is even given the name of the bodily faculty it is working with. Nevertheless, Descartes began his discussion of the cognitive power by claiming that it must be conceived as “purely spiritual” and “distinct from the whole body.” How, then, is this claim to be understood?

Concerning the word “spiritual,” Descartes gives very little indication in the Rules of what he means by it. Other than his use of “spiritual” to characterize the cognitive power, related words are only used two times in this work, but again without any direct indication of their meaning. 36 It seems clear from the context of these other two instances, however, that the

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36 See the entries for “SPIRITUALIS” and “SPIRITVS” in J.-R Armogathe and J.-L. Marion, Index Des Regulae Ad Directionem Ingenii De René Descartes avec des listes de leçons et conjectures établies par G. Crapulli (Rome: Ediziono dell’Ateneo, 1976), 127.
word is used as equivalent—or at least nearly equivalent—to “intellectual”. The first use of “spiritual” in the Rules comes in Rule 8, where Descartes divides the class of simple natures into the further sub-classes of “spiritual, or corporeal, or pertaining to both” (399); in Rule 12, however, when Descartes returns to this distinction to spell it out more fully, he divides the simple natures into the “purely intellectual, or purely material, or common [to both]” (419). When he then goes on to delimit the domain of simple natures common to both the intellectual and the material, he describes it as “that which is attributed without discrimination sometimes to corporeal things, sometimes to spirits” (419). These two other uses of “spirit”-words in the Rules thus seem to indicate an equivalence with “intellectual” in that they are both instances in which the two words are used interchangeably. Furthermore, the use of “pure” to modify “material” in the quote above indicates that in this context “pure” means something like “completely” or “thoroughly.” To say that the cognitive power is purely spiritual, then, would seem to mean simply that it is entirely intellectual. This, however, is confusing, as Descartes has made it clear that the cognitive power operates at a complete remove from the body in only one of its functions and only then is it called the pure intellect. It would seem to be unwarranted, then, to conclude from the cognitive power’s purely spiritual characteristic that it exists completely independently from the body.

What about its being characterized as “distinct” from the body? Note first that there have been no ontological or metaphysical considerations brought to bear on any of these matters. Descartes’s distinction between the cognitive power and the body comes in the course of his methodologically self-supporting account of cognition, an account of which he has repeatedly

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37 Descartes gives the following examples of the various kinds of simple natures: purely intellectual - knowledge, doubt, ignorance, volition; purely material - shape, extension, motion; common to both - existence, unity, duration.
emphasized the utility regardless of its truth. Moreover, it is now clear that that distinction is made solely in terms of function or operation, as it is only in one of its capacities that the cognitive power shows its isolation from the body. The account thus seems to be neutral to the question of the ontological separation of the cognitive power from the rest of the body. The very verb, *concipere*, that Descartes has used throughout this discussion in its gerundive form to direct his reader, is defined within this account to mean “to form new shapes or ideas in the imagination.” This would seem to indicate that all he intends to give in this account is a new imaginative portrayal of the knowing faculties, rather than a true understanding of their nature. Even the definition of *ingenium* given above supports this, as it clarifies the overall goal of the *Rules*: in laying out rules for the direction of the natural intelligence, Descartes is aimed at helping the cognitive power form or attend to new ideas or images in the imagination.

According to the account of Rule 12, therefore, the cognitive power is distinct from the body only methodologically or operationally, in that it can be regarded as functioning distinctly from the body, while no indication is given that it can actually exist distinctly. The very comparisons Descartes gives are in line with this. The cognitive power was said to be no less distinct from the body than blood from bone and hand from eye, but those examples are made up of entities that also do not exist independently. Rather, both of those pairs consist of subordinate parts that are contained in a larger living whole. While those parts are genuinely distinct, even spatially so, they give no indication of ontological separation. In fact they entail the opposite: the ontological dependence of parts on the substantial whole that contains them. Thus there is no

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38 Later in Rule 12, Descartes gives as an example of a necessary connection the proposition “I understand therefore I have a mind distinct from body” (422). There is no exposition or argument given for this proposition, however, and accordingly there are no adequate grounds for reading “distinct” metaphysically or ontologically. The question of different kinds of dualism is precisely the question of the way in which mind and body are distinct.
indication in Descartes’s account that the distinction between the cognitive power and the body is one of separate entities; the only clear distinction between the two comes in one operation of a faculty that is otherwise connected with the body and seems to be an equally-subordinated part within a larger whole.

With this discussion of the various functions of the cognitive power Descartes has laid out the two sides of the mind-body divide, which might more appropriately be called the divide between the body and the purely intellectual operation of the cognitive power. While it is now clear that this divide is methodological or operational, as opposed to ontological or metaphysical, it is nevertheless still significant in that it has been said that the intellect alone is capable of truth and understanding. This establishes the difficulty and importance of the question of how the purely intellectual operation of the cognitive power can work in tandem with its bodily operations to know the external world, as truthful understanding would otherwise be confined to the purely intellectual realm. In other words, the operational or methodological separation of mind from body is enough to establish the epistemological problems that come from a mind-body divide. For even if it is only in one function that the cognitive power operates separately from the body, given that that function plays a privileged role in all knowing, the mind’s ability to know the external world is still very much called into question. There remains, then, a very serious divide that needs to be bridged.

As soon as this divide is established, however, Descartes begins preparing his attempt to overcome it. He does so by finishing his discussion of that which pertains to the knower in the act of knowing by delineating the aid that the cognitive power can receive from the other faculties, whereby he sets up the topic of the following rules. On the one hand, he says, if the
intellect is dealing with something incorporeal or in which there is nothing similar to the corporeal, it can receive no aid from the other faculties; in fact, the other faculties would only hinder the purely intellectual operation of the cognitive power and must therefore be kept at bay. If, on the other hand, the intellect is examining “something which can be referred to body, the idea of it must be formed as distinctly as possible in the imagination,” while “the thing itself [res ipsa], which the idea represents, must be exhibited to the external senses” (416-17). Yet, given that one frequently needs to attend only to certain aspects of a thing, other aspects should be disregarded so as to more easily facilitate the memory of the pertinent aspects. Accordingly, it is better if abbreviated figures (compendiosae figulae) are presented to the external senses, rather than the things themselves, and the briefer (breviores) these figures the better, so long as they are not so brief as to causes lapses in the memory.

This discussion of the aid the cognitive power can receive from the bodily faculties has a clear connection with the division between different classes of simple natures. When describing the purely intellectual simple natures, Descartes says that they “are known by the intellect through a certain innate light and without the help of any corporeal image,” for “it is not possible to form any corporeal idea which represents [them] to us” (419). This clearly corresponds to the situation in which the intellect can receive no help from the bodily faculties in its activity of understanding but would rather only be hindered by them. Thus, when the intellect is concerned with purely intellectual simple natures, it must avoid any engagement with the imagination and the senses. By way of contrast, the purely material simple natures (such as shape, extension and motion) exist only in bodies, while those simple natures which pertain to both (such as existence, unity, duration) are attributed to both bodily and intellectual things. Anything falling into either
of these two categories obviously can be referred to body, and thus when dealing with such things the intellect can and should receive support from the bodily faculties by depicting them in the imagination while also displaying to the senses the thing itself or an abbreviated representation of it.

From this it is clear that any time the cognitive power is concerned with the corporeal realm the intellectual and bodily faculties must work together in order to have knowledge. In this way, Descartes prepares the central question of Rule 14, namely, how the intellect and imagination can work together to facilitate the attainment of knowledge of bodily things. In discussing this question, then, he is directly aimed at the issue of how the intellect can know the external world. Yet it also clear that any attempt Descartes makes to address this question is predicated on the existence of the operational divide between the intellectual and corporeal faculties in that any such attempt must consist precisely in the overcoming of the divide between the two.

§ 6. The Establishment of the Mathematical Character of Rules 13-24

Before moving on to Rule 14, Rule 13 and the establishment of the mathematical character of all that follows must be addressed. This actually begins in the final paragraph of Rule 12, where Descartes divides “whatever can be known” into simple propositions and questions (428). The former, which are known through intuition, cannot be sought out but rather occur to us spontaneously (sponte). This, he says, has been covered in the first twelve Rules. The latter are divided into perfectly and imperfectly understood questions, of which the first are
to be covered in Rules 13-24 (this section of the text exists only partially), while the second are to be covered in Rule 25-36 (this part to of the text is completely non-existent).

In Rule 13, Descartes says that the domain of questions is “all that in which the true and false is found” and thus that something only becomes a question when “we decide to make some determinate judgment about it” (432). He fleshes out his understanding of perfect and imperfect questions by listing three aspects common to all questions: first, there must be something unknown, otherwise there would be no point in asking the question; second, the unknown must be designated (designatum) in some way, otherwise the investigation wouldn’t be determined to that unknown in particular; and third, the unknown must be designated by way of something else that is known. While these three characteristics hold true of both perfect and imperfect questions, what distinguishes a perfectly understood question is that it is “so determined in every way that nothing more is sought than what can be deduced from the givens” (431).

Back in Rule 12, Descartes lists three things that must be distinctly perceived in order for a question to be perfectly understood, namely, the signs (signis) by which it is possible to recognize what is being sought when it occurs, what exactly the sought is to be deduced from, and how it can be proven that the given and the sought are so mutually dependent that the one cannot be altered without changing the other. Despite being perfectly understood in this way, the solution to such a question is unknown, yet all the necessary premises are present and nothing remains to be shown “other than how the conclusion is to be found . . . by unfolding some one thing dependent on many mutually implicated things” (429). Such questions, Descartes says, tend to be abstract and occur almost exclusively (fere tantum) in arithmetic and geometry.
The mathematical character of perfect questions is then echoed in Rule 13. That rule itself states that in order to understand a question perfectly, it must be abstracted from all superfluous conceptions, reduced to its simplest form, and divided up into its smallest parts with enumeration. By following this rule, Descartes says, one can reduce a well-understood problem to the point that “we think we are no longer concerned about this or that subject, but rather only with certain magnitudes in general [in genere circa magnitudines quasdam] which are to be compared with each other” (431). All perfect questions, therefore, reduce to proportional relationships between magnitudes, and thus the answering of all perfect questions is a matter of solving for the unknown in such proportions.

Descartes goes further, taking the examples of investigating the nature of magnets and sound, and claims that all imperfect questions can be reduced to perfect ones. In making this claim, Descartes suggests that anything beyond simple propositions can ultimately be reduced to a question of the relationship between magnitudes. This is a key moment in his attempt to reduce all scientific investigation to a mathematical form. Given that what remains of the Rules is said to be concerned with perfect questions, it is implied that everything that follows is connected with this goal. In fact, Rule 14 is aimed precisely at the issue of how to deal with questions in terms of the relationships of magnitudes, whereby it makes a major contribution to Descartes’s goal of giving the sciences a mathematical character. As will be seen in the next section, this aspect of Rule 14 turns out to be intimately connected with the attempt to overcome the mind-body divide such that knowledge of the external world is possible at all.
§7. The Account of Mathematical Cognition in Rule 14

Rule 14 reads as follows: “At the same time, it must be transferred to the real extension of bodies and be presented to the imagination entirely by bare figures [nudas figures]: for in this way it will be perceived by the intellect more distinctly” (438). Descartes leaves the subject of this sentence unexpressed, but it is presumably any question that one is dealing with. The claim, then, is that the question should be reformulated in terms of extension and presented to the imagination as simple extended figures, as this will allow the intellect to handle the question more easily. It is thus immediately clear that Rule 14 is concerned with that domain of questions in which the intellect is aided by the imagination, namely, questions that concern the corporeal realm, and that those questions are to be dealt with mathematically.

Descartes begins the body of Rule 14 by arguing for the reduction of all questions to the relationships of magnitudes. The key premise of this argument is that “in all reasoning [ratiocinantione] it is through comparison alone that we recognize the truth precisely” (439). The importance of comparison, Descartes argues, can be seen from the following. When an unknown is deduced from a known, the unknown must participate in the same nature as the known. Accordingly, deduction is merely an extension of the knowledge of an already known nature, rather than the discovery of “some new kind of entity” (438). The recognition of an already known nature in diverse subjects, however, occurs by transferring the same idea from one subject to another “by means of a simple comparison, through which we affirm that the sought is similar, or the same, or equal to something given in one respect or another” (439). Descartes thus advises his reader to conceive all knowledge other than that gained through the pure intuition of an individual to be obtained through the comparison of two or more things.
Such comparisons must be simple and overt (simplices et aperta) if their truth is to be intuitable by the light of reason alone, and this is only the case when the sought and the given participate equally in their shared nature. If the common nature is not shared in equally, it is present only according to some relation or proportion (habituidines sive proportiones) and the comparison is thus in need of preparation. In fact, Descartes says, “the chief part of human industry consists in nothing other than reducing these proportions to the point that the equality between the sought and something that is known is clear” (440). The only thing that can be reduced to such an equality is that which admits of more and less, namely, that which is comprehended by the word “magnitude” (magnitudinis). Thus, Descartes concludes, having abstracted a question from every subject as Rule 13 teaches, one is left to deal only with magnitudes in general (magnitudines in genere).

This is Descartes’s argument for the reduction of all questions to the relationship between general magnitudes, although he has not yet given any indication of how such magnitudes are to be understood. It will eventually become clear that these general magnitudes are abstractions existing only in the pure intellect, which is evident in what immediately follows but not made explicit until later in Rule 14.

Descartes next begins to lay out how the imagination should be employed in the service of the intellect such that the solution of all questions can be found by operating with the real extension of bodies. Here the essential relationship between the intellect and imagination in all mathematical cognition becomes clear. He begins by saying the following: “But in order that we might imagine something even then, and not use the pure intellect, but the intellect aided by images [speciebus] depicted in the fantasy, it must then be noted that nothing is said about
magnitudes in general that cannot also be referred to any species of magnitude” (440-41). It will therefore be of “no small profit” to transfer what is understood about magnitudes in general to that particular species of magnitudes that is most easily and distinctly depicted (plingetur) in the imagination, namely, “the real extension of body [extensionem realem corporis] abstracted from everything other than that it is shaped” (441). That this is the species of magnitude most easily depicted in the imagination is clear from the bodily nature of the imagination; as a corporeal organ, the impressions it contains are nothing but extended figures. Moreover, the real extension of bodies is the best subject for distinctly and exactly exhibiting the differences of all proportions, as evidenced by the fact that all exact determinations of proportion are accomplished by means of an analogy with extension. Descartes thus concludes, “It remains firm and fixed, therefore, that perfectly determinate questions contain hardly any difficulty other than that which consists in developing proportions into equalities, and that everything in which precisely such a difficulty is found can easily be, and ought to be, separated from all other subjects and then transferred to extension and figures” (441). He then indicates again that this alone is the subject matter of Rules 13-24.

Here Descartes has extended his reduction of all questions from the relationship between general magnitudes to the proportions of extended magnitudes depicted in the imagination. The reason he takes this step is that once a question has been narrowed down to the relationships of magnitudes in general, “even then” something needs to be present in the imagination so that the pure intellect is not operating on its own, and general magnitudes themselves cannot be present in the imagination because of its bodily nature. So in order to use the intellect in conjunction with the imagination—as the nature of such questions calls for and as is necessary to keep the
pure intellect from error—the imagination must contain some specific species of magnitude
which it presents to the intellect, while the intellect itself is concerned with general magnitudes.
In this way, the general or abstract concept is distinctly present to the intellect, while an image of
it is present in the imagination.

With this Descartes indicates the importance of the distinction between general and
particular magnitudes. While he skips over what is at stake in this distinction without making it
explicit, the central issue here is the determinacy of particular magnitudes versus the
indeterminacy of general magnitudes. The pure intellect can only be concerned with
indeterminate magnitudes, while the imagination can only be concerned with determinate
magnitudes. This is precisely because of their being on opposite sides of the mind-body divide.
For it is the bodily specificity of a magnitude that makes it determinate, while the general idea of
magnitude has no specific determinacy but is rather an abstraction that can exist only in the
intellect. How such an abstraction arises will be considered in what follows. It is already clear,
however, that the approach Descartes is advocating in Rule 14 requires the coupling of an
abstract concept in the intellect with a determinate representation in the imagination.

From what has been considered so far of Rule 14, it is clear that Descartes is sketching
out a mathematical way of dealing with all questions that concern the material realm by reducing
them to the relationships between magnitudes. In the rest of Rule 14, he goes on to solidify and
specify the mathematical character of this approach. He begins, however, by emphasizing that
the mathematical aspect of his method is aimed beyond the purely mathematical realm at “a
more profound wisdom” (altiorem sapientiam—442). Although he does not say what this
wisdom is, it presumably consists of mathematically-bolstered knowledge of the corporeal
world. While Descartes here articulates a clear subservience of mathematics to method, he nevertheless spends the rest of Rule 14 clarifying his understanding of various mathematical terms (most notably “extension,” but also “dimension,” “unity,” and “figure”), arguing that they must be understood in connection with bodily subjects. He thus makes clear that a proper understanding of the mathematical aspect of his method is essential to his endeavor of gaining knowledge of the corporeal world.

By “extension,” Descartes says he means simply that which has length, breadth and depth. According to him, nothing is more easily perceived by our imagination than extension and thus no further explication is needed. Nevertheless, he then goes on to give a lengthy polemic against the position of the “learned” (literati) on this matter, and it is in this discussion that he expounds his understanding of the nature and origin of the abstract ideas in the pure intellect, which he sets up in contrast to an obscure and false understanding of abstraction that depends upon a false judgment of the intellect.

As opposed to the learned, Descartes refuses to recognize “philosophical entities that do not actually fall under the imagination,” and he therefore holds that “extension does not designate something distinct and separate from the subject itself” (442). According to him, one can see that it is a mistake to believe that extension itself can exist on its own, separate from all extended things from the fact that it is impossible to form an image or idea of extension in the imagination without some extended subject underlying it. When the intellect takes such an abstract entity (ens abstractum) to exist separately from its underlying subject, it makes an incorrect judgment that cannot be brought into line with the imagined idea. Such an understanding depends precisely upon that mistake that Descartes has set out to avoid, namely,
the intellect’s inappropriately acting on its own without the aid of a corporeal idea in the imagination. As he is not concerned with anything that cannot be considered without the aid of the imagination, a separately existing abstract entity is simply of no interest to him.

It is therefore important, he says, “to distinguish carefully those ideas by means of which the individual meanings of words are to be presented to the intellect” (443). In order to further clarify the meaning of the word “extension” and the corresponding idea in the imagination, he then considers three sentences: “extension occupies place,” “body has extension,” and “extension is not body.” The first shows that “extension” may be taken to mean “that which is extended,” and there is in fact no difference in conception between those two locutions, although they should not simply be used interchangeable as that might obscure what is being conceived in this sentence, namely, that “some subject occupies place because it is extended” (443). In the second sentence, “body has extension,” “extension” signifies something other than body, but nevertheless there is only one idea in the imagination, namely, that of an extended body; in terms of the thing itself “body has extension” is no different from saying “body is extended” or even “the extended is extended.” This is because “extension” is an entity that “exists only in something else, and which can never be conceived without a subject,” as opposed to that kind of entity “that can really be distinguished from its subject” (444). It is failing to make this distinction that leads to the false opinion that extension is something other than that which is extended.

In the third sentence, “extension is not body,” “extension” is meant differently than in the previous two sentences, as it can have no corresponding idea in the imagination where extension and body are inextricably joined. Instead it is merely an abstract entity that is “separated out” by
the pure intellect. Such entities cannot be comprehended by the imagination, and those who try to represent it in the imagination make many errors, such as contradicting themselves in saying that extension is not body while depicting it by means of an idea that necessarily involves body. It is thus very important, Descartes says, to distinguish those terms and propositions that “have such a restricted meaning that they exclude something from which they are not really distinct” as they must be “entirely removed from the imagination in order to be true” (445). He then gives the following examples—all mathematical—in which the terms have such a restricted meaning: “extension or figure is not body, number is not the things numbered, a surface is the limit of a body, a line is the limit of a surface, a point is the limit of a line, and unity is not a quantity” (445). The terms in these propositions are conceived in such a way that they exclude that from which they are not really distinct and thus they cannot come before the imagination. Accordingly, they have no place in Descartes’s mathematical investigations.

Yet these terms can be used while retaining their abstracted meanings but without excluding or denying that from which they are not distinct. In such cases the imagination can and should be used to aid the intellect so that it is not led astray. For although the intellect is attending “solely and precisely” to the abstract meaning of the term, the need may arise to attend to the other features of the thing itself that are not expressed by the term. In that situation, if the imagination contains a “true idea of the thing” that the intellect can consider, it will be prevented from making an incorrect judgment about the exclusion of those features.
Descartes gives some examples that make this clearer. When concerned with a question about number, while the intellect is reflecting on “mere multitude” (*solam multitudinem*), the imagination should imagine some subject measurable by many units so that the intellect will be prevented from making the mistaken judgment that number excludes the things numbered, which leads to attributing “miraculous and mysterious” properties to numbers that are actually “sheer nonsense” (*meras nugas—445*). Similarly, when concerned with figure one is really dealing with an extended subject that is conceived only insofar as it has shape; when concerned with a body one is really dealing with that which has length, breadth and depth; when concerned with a surface, one is dealing with the same thing, but only conceiving of it in terms of length and breadth, omitting depth but not denying it has depth; when concerned with a line, one is again dealing with that which has length, breadth and depth but only conceiving of it insofar as it has length; and finally, when concerned with a point, one is dealing with a subject from which everything has been omitted except that it exists.

Descartes does not spell out what is in the intellect versus the imagination in these latter examples, but an analogy with the example of number makes it clear. In the case of figure, the pure intellect would be concerned with an abstract entity that could be called “mere figurality” or just “figure itself,” while the imagination would contain an idea of some particularly shaped extended subject. In the case of the other examples, the pure intellect would be concerned with the abstraction of various features of an extended body while it ignored all others, and the

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39 Klein glosses this as “‘multitudinousness’ as such” (“Mengenhaftigkeit* als solche*) which he goes on to describe as “the ‘naked’ indeterminate manyness to which simply nothing ‘true,’ nothing truly in ‘being,’ and hence no ‘true idea’ of a being corresponds”; Klein, *GMTOA*, 201.
imagination would contain an image that might highlight that particular feature but not exclude other necessarily connected features.

This amounts to Descartes’s account of the origin of abstract ideas as well as their proper use. The pure intellect separates a general, abstract entity from the subject it is necessarily contained in, which it then considers by itself. In doing so, it excludes that from which it is not actually distinct, which is not problematic so long as the intellect simply omits or ignores those other features rather than making a judgment that denies their necessary connection. The intellect is prone to making such a mistaken judgment, however, so in order to avoid that the imagination must aid the intellect by presenting a true image of the thing itself. Descartes is thus not opposed to abstraction per se, but he clearly thinks the improper use of abstract ideas has caused many mistakes and confusions throughout the history of mathematics and philosophy, which he here aims to correct by teaching the proper way to handle such abstractions.

How, then, is this account applied to extension, which is Descartes’s main concern in this portion of the text? In that case, the pure intellect would separate out the extendedness from an extended subject and then focus on that abstract entity by itself. While that abstraction could be called “mere extension” or “extension itself,” it is what Descartes referred to earlier as “magnitude in general,” noting that the real extension of bodies is just that species of magnitude that is most easily depicted in the imagination. While the intellect is concerned with this abstract entity, the imagination must contain some idea of an actually extended body, as this will keep the

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40 According to Klein, when the pure intellect is considering a general concept as its object, it is directed at its own act of knowing, whereby the intellect’s act is turned into an object insofar as that act has become the focus of the intellect’s own cognitive regard (see Klein, GMTOA, 200 & 208). While this is not explicit in the Rules, it can be seen to be the case from the account of cognition at work there. For insofar as the pure intellect is directed at an abstraction it is directed at a product of its own intellective activity.
intellect from making the mistaken judgment that extension or magnitude can exist separately from any extended, magnitudinous subject. As Descartes goes on to say, “here we are concerned with an extended object considering absolutely nothing else in it except extension itself” (447).

He then reminds the reader that he is here concerned with questions that have been reduced to the point that nothing is being sought but a certain unknown extension that is to be found by comparing it with already known extensions. Although Descartes does not spell it out himself, it is now clear what is going on in the cognition that is behind the reduction of any problem to the relationships between magnitudes or extensions. Given that “all the differences of proportions that exist in other subjects can also be found between two or more extensions” (447), once all the superfluous aspects of a question have been abstracted from—as demanded by Rule 13—the question can then be formulated in terms of a relationship or proportion between magnitudes that simply needs to be reduced to an equality. At this point “we no longer think that we are concerned about this or that subject, but rather only with certain magnitudes in general” (431). These general magnitudes are abstract entities that the intellect has separated out from the subjects they are contained in. The intellect then considers these general magnitudes in themselves as though they were separate from their subjects, even though they are in fact not. In order to keep the intellect from operating on its own, whereby it runs the risk of making mistaken judgments, the imagination must be called to the aid of the intellect, which it provides by presenting an image or idea of magnitude that does not exclude that which it is necessarily connected with. In doing so, it is best to employ “that species of magnitude that is most easily and distinctly depicted in our imagination,” namely, “the real extension of body abstracted from everything else other than that it is shaped” (441). In this way, the abstract general magnitudes
in the intellect are coupled with concrete determinate extensions in the imagination. The solution to the question can then be found by examining the relations and proportions that exist between these extensions and reducing them to the point that there is an equality between what is sought and some already known extension.

This is the mathematical cognition of Rule 14. The goal of this cognition, as is now abundantly clear, is to reduce all questions that concern the corporeal realm to a form in which bodily objects are treated mathematically. The concern with knowledge of the corporeal realm was already clear from Descartes’s focus on the domain of questions that requires the intellect to be aided by the imagination, which Rule 12 established to be the province of questions about the external, physical world. This is taken a step further in Rule 14, however, when Descartes excludes from his considerations all philosophical entities that cannot come before the imagination. In doing so, he definitively removes everything non-corporeal from his investigation; for insofar as the imagination is a bodily organ it is only corporeal ideas that can exist within it.

The mathematical aspect of Descartes’s approach was evident from Rule 14’s goal of reducing all problems to the point that they are concerned only with that species of magnitude that is most easily compared and manipulated, namely, the real extension of body. Beyond this, Descartes’s polemic against the learned shows that his account of the cognition that underlies mathematics entails a critique of the understanding of the mathematical sciences as concerned with independently existing objects. In opposition to that understanding, Descartes lays out an account which emphasizes that all mathematical objects should be understood in conjunction with corporeal subjects, and it is this account that provides the guidelines for the appropriate use
and understanding of mathematical objects. While it is true that the pure intellect is concerned with abstract entities in this cognition, those abstractions are nothing but certain aspects of a body that have been lifted off and considered in themselves; hence the requirement that the abstractions of the intellect always be accompanied by corporeal images in the imagination. In this way, even the mathematical aspects of the approach advocated in Rule 14 are intimately connected with the corporeal world, as is only appropriate given Descartes’s aim of teaching how to employ mathematics in the pursuit of knowledge of the corporeal world. It can thus be said that Descartes collapses the distinction between the mathematical and the physical in that he not only treats corporeal object mathematically but also makes all mathematical objects dependent on corporeal ones.

It must be emphasized, however, that the abstractions employed in this mathematical cognition, although rooted in corporeal subjects, are significantly different from their corresponding ideas in the imagination. As independently considered abstract entities proper to the pure intellect, they are general and indeterminate in a way that no bodily subject can be. This is central to the mathematical approach of Rule 14, which calls for the coupling of such an indeterminate abstraction, namely, magnitude in general, with a determinate extension. This coupling is a key step in the invention of the modern symbolic concept of number (as discussed in §7 of Chapter 1), and Descartes spends the rest of the extant Rules trying to show how to operate with the new mathematical object it gives rise to.41

41 Klein’s discussion of Rule 14 and its relationship to his analysis of the invention of the modern symbolic concept of number will be discussed in the following section. In the section after that, Descartes’s attempt in Rules 15-21 to explain how to calculate with this new concept of number will be considered briefly, although a full discussion of this matter is postponed until Chapter 3’s in-depth consideration of the Geometry.
It is now clear, therefore, that the mind-world separation is not only built into the mathematical cognition of Rule 14, in that it hinges upon the relationship between the intellect and the imagination, but also that the bridging of the gap between those two faculties is responsible for shaping the very nature of that cognition and the object considered in it. For that object is created by the combination of a general, indeterminate abstraction, which is proper to the pure character of the intellect, with a concrete, determinate extended subject, which is proper to the bodily nature of the imagination. Descartes’s mathematical cognition thus depends upon bringing together two elements from opposites sides of the mind-body divide while maintaining their importantly different characteristics. It is in this way that the operational or methodological dualism of the Rules is essentially built into its account of mathematical cognition.

Having completed the argument for the reduction of all questions to the comparison of extended magnitudes and explained how to go about using the cognitive faculties for this purpose, Descartes concludes Rule 14 with a brief discussion of “all those features in extension itself that can aid in expounding differences of proportions, of which there are only three, namely, dimension, unity and figure” (447). In doing so, he sketches out key aspects of the procedure for actually employing the mathematical cognition laid out above.

By “dimension,” Descartes means “nothing other than a mode or aspect according to which some subject is considered measurable,” such as length, breath, depth, weight or speed (447). There is an infinite number of such modes or dimensions in any given subject, he says, some of which are “real,” being rooted in the subject itself, while others are merely “intellectual” in that “they have been thought out by a judgment [arbitrio] of our mind” (448).
“Unity” is the common nature that is equally shared in by everything that is to be compared together. So long as a unit has not yet been determined in a question, any magnitude can be adopted as the unit, which will then serve as the common measure for all the rest. In order to be useful, however, the unit must be understood to contain as many dimensions as there are in the extreme terms that have to be compared with each other. Such a unit can be conceived either as a point, a line, or a square.

Concerning “figure,” Descartes first reminds the reader that it has already been established that “the ideas of all things can be formed [fingi] by figures alone” (450). He then says that he is only going to use those figures that most easily express all the differences of relations and proportions, which turn out to be straight lines and rectilinear figures. He lays out his reasons for using these two figures alone as follows. There are only two kinds of things that are compared with each other, multitudes and magnitudes, and there are two corresponding kinds of figures, namely, discrete and continuous ones. Moreover, all relations fall under one of two categories, order and measure; in an order, the parts are related to each other immediately, in terms of themselves alone, while in a measure they are related by a mediating third term. It is always possible, however, at least in part and sometimes completely, to reduce a continuous magnitude to a multitude by selecting an appropriate unit, whereby “the difficulty pertaining to the knowledge of a measure ultimately depends solely upon an inspection of the order”; in fact, Descartes says, “the greatest aid of the art lies in this progression” (452). Thus by means of

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42 This appears to be a reference to Rule 12, where it is said that “it is certain that the infinite multitude of figures suffices for expressing all the differences of sensible things” (413), although no argument for that claim is made there or elsewhere in the extant Rules.

43 Descartes gives two examples of figures used to represent multitudes: six dots arrayed in a triangle and a family tree consisting of “father,” “son,” and “daughter” (450). The oddity of the latter is worth noting, as it highlights a tension in the reduction of all relations to the relations of quantities.
continuous figures both magnitudes and multitudes can be exhibited and compared, while the choice of a unit allows the relations involved to be reduced from one of measure to one of order. Furthermore, it is a rule of the art that only two things are to be compared at a time, and length and breadth are the dimensions of a continuous magnitude most distinctly conceived. These two dimension alone, therefore, shall be attended to, as “nothing more simple can be found by human industry for expressing all the differences of relations” (452).

These three aspects of extension—dimension, unity and figure—thus work together to show how the comparison of magnitudes can be understood and treated by simply choosing a unit to measure whatever dimension or aspect of a body is being compared and then depicting the relations or proportions between those magnitudes by means of the most convenient figures, namely, lines and surfaces. Descartes has here set up a number of fundamental aspects of his innovative mathematics, such as the arbitrariness of the unit and the collapse of the distinction between multitude and magnitude. The discussion of these three aspects thus serves as the beginning of Descartes’s attempt to bring his account of mathematical cognition to bear on the development of actual mathematical practices. Before examining the continuation of that attempt in the remainder of the Rules, however, a brief return to Klein’s discussion of symbolic abstraction is called for.

§ 8. Klein on Symbolic Abstraction and Rule 14

The mathematical cognition of Rule 14 is, on Klein’s account, the symbolic abstraction that gives rise to the modern symbolic concept of number. As discussed in §7 of Chapter 1, the modern concept of number is symbolic in Klein’s sense in that it involves the conflation of a
second intentional object with the first intentional sign used to represent it. In §6 of Chapter 1, it was shown that a first intention always intends or refers to a specific and determinate object, while a second intention intends a general and indeterminate one, which must therefore be an ens rationis—a being that exists only in and because of the mind. A symbol thus inherently involves the combination of the indeterminacy of a general object in the mind with the determinacy of a particular sign used to represent it.

In the mathematical cognition of Rule 14, the second intentional object is the abstract concept considered by the pure intellect, which has indeed been shown to be general and indeterminate, while the first intentional object is the bodily depiction in the imagination that is coupled with that abstraction. In this way, the imagination provides a symbolic representation of the indeterminate content that has been separated out by the pure intellect from a determinate object.

It is clear from Descartes’s examples throughout Rule 14 that this process allows for the generation of a range of symbolic constructions within the mathematical realm. Klein is aware of this, as is clear from his brief discussion of the symbolic understanding of figure, but he is primarily interested in the symbolic concept of number. In that particular case, the object of the pure intellect is some general magnitude, such as the mere multitude of five-ness, while the determinate image used to stand for that object is the corporeal extension in the imagination, such as a line-length with five unit-lengths marked off on it.

44 See Klein, GMTOA, 299 n. 319.
45 As will be considered shortly, the figural representations of the imagination can be externalized and displayed to the senses or simply replaced with algebraic letter signs.
The full conflation of the general object and its representation, however, only occurs when the imaginative depiction is used to represent the indeterminate abstraction in the course of actual mathematical operations, whereby the indeterminate object of the pure intellect is treated determinately. The complete process of symbolization thus occurs only in the actual employment of this new concept of number in the operations of mathematical practice. In this way, Rule 14 only lays the groundwork for the possibility of symbolization by providing an epistemological account that underlies the process of conflating an indeterminate object with its determinate representation. A step beyond such epistemological considerations into the realm of mathematics is required for the full-blown symbolic concept of number. While Descartes spends the closing portion of the extant Rules trying to sketch out this final step, it is only in the Geometry that he brings it to full fruition.

At this point, however, it is already clear that insofar as the account of mathematical cognition in Rule 14 lays out the epistemological basis that underlies the process of symbolization, that account is central to the very possibility of the symbolic concept of number. Moreover, it is now also clear that Descartes’s non-substantial dualism is central to that account in that the operational remove of the mind from the world underlies the very possibility of symbolic abstraction. In this way, then, the understanding of the mind-world relationship found in the Rules is built into the epistemological account that underlies the symbolic concept of number, whereby it is also built into that concept itself. 46

46 As mentioned in Footnote 85 of Chapter 1, both Klein and Hopkins claim that Descartes’s account is the only account that has been given of the cognition that underlies the use of symbolic concepts (see Klein, GMTOA, 295-96 n. 314, and Hopkins, Origin, 501-02). It was also mentioned there, however, that the work of Klein and Hopkins allows for a phenomenological account of symbolic abstraction that does not depend upon all the details of Descartes’s account (see Hopkins, Origin, 526-28). To now make this somewhat more specific: the alternative
§ 9. The Mathematics of Rules 15-21

As already mentioned, Descartes spends the rest of the extant Rules working out the employment of the mathematical cognition of Rule 14 in the actual operations of mathematical practice. The attempt to sketch this out, which consists of Rules 15-18 and the enunciation of Rules 19-21, only amounts to a first beginning that appears to have been broken off when Descartes stopped working on the Rules. Nevertheless, these closing Rules provide a preliminary indication of the dependence of Descartes’s mathematics on the epistemological account laid out in Rules 12-14. The following overview of these Rules thus serves as an anticipation of what will later be seen in the Geometry.

Rule 15 teaches that “it is generally helpful to draw figures and exhibit them to the external senses, so that by this means our thought is more easily held attentive” (453). In the body of Rule 15, Descartes outlines various ways, depending on the situation, to visually depict the unit as well as the comparison of magnitudes. The unit can be depicted by a dot, line or square, although it must always be understood to be extended and thus infinitely divisible; the comparison of magnitudes can then be depicted by positioning them in relation to that unit. Throughout the brief discussion of this topic, the emphasis is on externalizing what is in the

phenomenological account of symbolic abstraction is similar to Descartes’s account in that it depends upon a distanced and self-reflexive directing of the intellect’s cognitive regard, but it differs in that it does not depend upon the account of a bodily imagination interacting with a disconnected intellect. This indicates that it is only Descartes’s operational remove of the intellect that is essential to symbolic abstraction, while the specifics of his underlying account of the faculties are not. This is important in regards to the conceptual presuppositions of modern symbolic mathematics, as it means that it is only the non-substantially dualistic mind-world relationship that is built into the nature of symbolic concepts, without any ontological commitments. Any discrediting or discarding of such doctrines as Rule 12’s account of the faculties (with its corporeal imagination) or Descartes’s later full-blown substance dualism thus has no bearing on this stratum of sedimentation.

47 For a historical reconstruction of the reasons that led Descartes to give up working on the Rules, see Schuster, Descartes-Agonistes, 334-44 (especially 343-44 where he discusses the difficulties presented by the mathematics of the final Rules).
imagination so that it is presented to the eyes, which Descartes claims to be a straightforward and obvious task given the underlying account of the bodily faculties. The aid offered to the intellect by the imagination is thus extended to and supported by the senses in a way that depends upon the account of Rule 12.

Rule 16 augments this sensory aid by teaching that it is better to “represent” (designare) by “very concise signs [brevissimas notas] rather than by complete figures” that which is necessary for the solution of a problem but does not currently require attention, as this will prevent the memory from being misled while also keeping the intellect from being distracted (454). To this end, Descartes advocates the algebraic practice of designating letters to express magnitudes and then combining them with number signs. This, in turn, is an essential part of a general mathematical procedure that is sketched out in the course of Rule 16. Emphasizing that in this procedure one abstracts from numbers themselves as well as from geometrical figures, Descartes says that one should begin solving a problem by abstracting from its specific terms “in order to investigate its nature” by “expressing it in general terms,” and that it is only after this that “it should be re-expressed in terms of the given numbers” (457). In outlining this procedure, Descartes makes it clear that what he is describing here is the process of reformulating a problem in general algebraic terms, which are then manipulated to give certain relationships that the given numbers can be plugged into in order to find the solution. In this way, Descartes attempts to extend the symbolic construction of Rule 14 through the use of algebraic practices.

In Rule 17, Descartes says that Rules 17-21 will teach how to treat any problem by subordinating unknowns to one another in a continual proportion beginning with the unit such that, regardless of the number of unknowns, a sum will be yielded equal to a known magnitude.
In this way, these Rules are intended to be a further explication of the procedure outlined in Rule 16. Rule 17 itself says that “[t]he proposed difficulty must be run through directly, abstracting from the fact that certain of its terms are known while others are unknown, and intuiting the mutual dependence of the terms on each other through true discursive reasoning” (459). Here Descartes advocates treating known and unknown terms in the same manner so that they can be related in the easiest and most direct way. In this too, it should be noted, he is drawing upon algebraic practices in that it is the use of algebraic signs that allows the knowns and unknowns to be treated equivalently.

Rule 18 says that only four operations are needed to deduce unknown magnitudes from known magnitudes, namely, addition, subtraction, multiplication and division. The first two are used when the magnitudes involved are directly related as parts contained in a whole. The latter two are used when the magnitudes are indirectly related, in which case a continuous proportion must be built out of the knowns and unknowns starting with the unit, which “is here the basis and foundations of all relations” (462). Beyond understanding how these operations work, however, it must also be explained “how [they] are to be presented to the examination of the imagination, and even how they are to be exhibited to the eyes themselves” (464). Descartes then goes on to lay out a way of representing these operations by means of lines and rectangles marked with unit lengths, whereby he attempts to show how algebraically expressed operations can be depicted by means of symbolic magnitudes.

Rules 19-21 exist only as enunciations; there are no explications, only the mere statements of the Rule. While their meaning is not exactly clear because of this, they seem to be
generally aimed at finding and handling equations (aequationes) which express the relationships between known and unknown magnitudes. This, however, is where the *Rules* breaks off.

For present purposes, the most important point concerning Rules 15-21 is that Rule 14’s account of mathematical cognition is clearly being put into practice in the mathematical procedures laid out therein. While these Rules represent only the beginning of an attempt to build a mathematics on this basis, Descartes is here moving towards the analytic geometry of his later mathematical thought.\(^{48}\) The following chapter will consist of an in-depth look at the *Geometry*, wherein it will be shown that the mathematical cognition of Rule 14, as well as the mind-world relationship upon which it is predicated, are still at work in Descartes’s mature mathematics.

\(^{48}\) For an account of the development of Descartes’s mathematical thought from the *Rules* to the *Geometry*, see Sasaki, *Descartes’s Mathematical Thought*, Chapters 4 & 5.
CHAPTER 3

The Concept of Space in Descartes’s Geometry

§ 1. Introduction to the Present Analysis of the Geometry

Descartes’s Geometry was published in 1637 together with the Optics, the Meteorology, and the Discourse on Method, the last of which served as an introduction to the other three. In Part II of the Discourse, wherein Descartes gives an overview of his method, he also gives the following description of his mathematical procedure:

Seeing that, although the objects of [all the particular sciences commonly called mathematics] differ, they still did not cease to accord with each other in that they consider nothing other than the diverse relations and proportions that are found there, I thought it would be more valuable to examine only those proportions in general (proportions en general), and to suppose them only in subjects that would allow me to know them more easily, but without restricting them in any way only to these subjects, so that later I could apply them all the better to all other things to which they might by suited. Then, having noted that in order to know them I would sometimes need to consider each in particular, sometimes only to retain them, or to comprehend several together, I thought that in order to consider them better in particular I should suppose them [to hold between] lines, since I found nothing more simple or that I was able to represent more distinctly to my imagination and my senses; but that in order to retain them, or to comprehend several together, it was necessary that I construe [expliquasse] them by certain ciphers [chiffres], as short as possible; and that by this means I would borrow all the best of geometrical analysis and algebra, and correct all the defects of the one by the other. (20)

This passage provides something of a bridge between the Rules and the Geometry. From the previous chapter’s analysis, it should be clear that what Descartes describes here is in line with

his views expressed in the *Rules*, indicating that at the time of the publication of the *Geometry* Descartes still thought it was appropriate to describe his general mathematical approach in a way that fit with what he had laid out in that earlier work. Insofar as this description also fits with the mature mathematical procedure of the *Geometry* (as will become clear from the present chapter’s analysis), this passage serves as an indication from Descartes himself that there is a connection between the *Rules* and the *Geometry*.

It is the first main goal of the present chapter to spell out this connection by observing the use of symbolic mathematics in the *Geometry* (§ 3) and demonstrating that this use is based upon the account of mind in the *Rules* (§ 4). This, in turn, will prepare us for the second main goal, which is to identify the concept of space at work in the *Geometry* (§§ 5-6) and demonstrate that it is itself symbolic (§ 7). Before taking up an analysis of the *Geometry*, however, we begin with a look at David Lachterman’s discussion of that work (§ 2), which serves as an introduction to our own analysis by way of providing an overview and anticipation of some of the important topics that will be fleshed out more fully in what follows.  

§ 2. David Lachterman on the Nature of Cartesian Mathematics

In his book *The Ethics of Geometry*, David Lachterman has drawn his own connection between the mathematics of the *Geometry* and the account of the mind in the *Rules*. On his view, “The whole of the second book of the *Rules* [i.e., Rules 13-24] seems to have been conceived as a prolegomenon to a presentation of Descartes’ new geometry, even if not precisely

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2 Let the reader be forewarned that this section is intended primarily as an introduction to the ontological and epistemological issues of the *Geometry*, while its technical details are not considered in depth until that work is taken up directly in the ensuing sections.
To understand this claim, however, one must first understand Lachterman’s general thesis concerning the nature of modernity and its essential relationship to modern mathematics.

According to Lachterman, the hallmark of modernity is “the ‘idea’ of construction or, more broadly, the ‘idea’ of the mind as essentially the power of making, fashioning, crafting, producing, in short, the mind as first and last poïëtic and only secondarily or subsidiarily practical and theoretical.” In this way, the mind is not a receptor of the world as it is, but rather the generative source of the nature that is imputed to the world. While Lachterman recognizes that this is not enough to characterize modernity completely, he believes that construction is the fundamental element that unifies the constellation of themes that make up modern thought. Yet he also recognizes that this is a rather broad claim that is difficult to substantiate, so he confines the goal of his book to uncovering modernity’s originary impulse toward construction, which he finds in Cartesian mathematics. In fact, it is part of his general thesis that the modern revolution not only begins in mathematics but continually takes its guidance from that starting point. As he says, the modern emphasis on construction is “principally the outcome of a signal alteration in the way mathematics itself is practiced and understood.”

From this it can be gleaned that Lachterman is working in the line of investigation initiated by Klein and Husserl. His awareness of Klein is well established, and his indebtedness

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4 Ibid., 4.
5 Ibid., viii.
to him would be difficult to deny. In addition to extending Klein’s work, however, he also intends his analysis as a deepening of it. This is clear from the “Preface,” in which Lachterman explicitly places the focus of Klein’s analysis in a subordinate position to his own: “Symbolization and formalization, while also obvious hallmarks of modern mathematics, need to be understood as supplements to, or variations on, the root idea of construction.” Nevertheless, Lachterman is willing to speak of uncovering the “‘sedimented’ character” of “Descartes’ founding gestures,” and in this way he can be understood as attempting to desediment a deeper stratum than Klein himself reached.

Lachterman’s attempt to deepen Klein’s analysis by emphasizing the role of construction extends to a discussion of Cartesian mind. In addition to symbolization, Lachterman understands the precondition “most germane to the possibility of Cartesian thinking” to be mechanization. This includes not only the mechanical understanding of the world and its operations, but also “the institution of the ‘mind’ as mechanically the most adept faculty” in that its proper use entails proceeding “mechanically in the fashion of an incorrigible computing device” which leads to

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6 While Lachterman does not mention Klein by name in the body of his text, he does cite him numerous times in his endnotes, always to a complimentary effect. Let it also be remembered that Lachterman is the translator of Klein’s lecture “The World of Physics and the ‘Natural’ World” in Lectures and Essays.

7 Carl Page, in his review of Lachterman’s book, states the situation succinctly: “Lachterman advances Klein’s point in two main ways: (1) he subordinates the insight into the symbolic . . . under his own enormously fruitful elaboration of construction as the essential ‘mark of the moderns,’ and (2) he pays far more attention to the cast of mind that both generates the new approach and is fired by its own success and daring”; Carl Page, “Mathematics and Modernity.” New Vico Studies 8 (1990): 64.

8 Lachterman, Ethics of Geometry, viii.

9 Ibid., 149. Lachterman echoes the language of Klein elsewhere in speaking of “Greek conceptuality” (125) and “symbolic abstraction” (186).

10 It is perhaps more appropriate to say that Lachterman aims at identifying rather than desedimenting this layer of meaning, as the sweeping nature of his thesis makes it difficult to establish definitively (a point Lachterman himself acknowledges throughout). The question of whether Lachterman is correct in his assessment that he is aiming at something more fundamental than Klein is a separate topic that can not be discussed here, although my inclination is to think that he is.

11 Lachterman, Ethics of Geometry, 125.
“the promise of endless inventiveness”; in fact Lachterman goes so far as to connect these two mechanistic understandings, saying “[t]he mechanization of nature advances pari passu with the machinations of the mind.”\textsuperscript{12} It is thus precisely this mechanistic understanding of mind that gives it the constructive power to be the creative source of the nature of the world. Moreover, as will be seen later, on Lachterman’s account this mechanistic view of the mind works together with symbolization in Cartesian mathematics, and in laying this out he adds a constructive element to Klein’s account of the way in which Descartes’s conception of mind underlies Descartes’s mathematics.

What, then, according to Lachterman, is the nature of Cartesian geometry? To begin answering this question, we must examine the contrast he establishes between it and ancient geometry, particularly as the latter is embodied in Euclid’s \textit{Elements}. The examination of this contrast is necessary because the invention of modern mathematics begins with an appropriation of ancient mathematics that amounted to a “fundamental metamorphosis” in its “conceptual and procedural understanding,” resulting in the “transfiguration of a theoretical into a productive or \textit{poiētic science}.”\textsuperscript{13} In addition to identifying the heart of the new understanding of mathematics as construction, Lachterman also identifies a number of other, connected, points of contrast between the two forms of geometry. Some of these concern the details of their mathematical practice, such as the fact that ancient geometry is primarily concerned with proving theorems (i.e., proving the truth of mathematical propositions) while Cartesian geometry is primarily aimed at solving problems (i.e., performing mathematical tasks), or that ancient geometry has strict homogeneity requirements governing the comparison of magnitudes, while Cartesian

\textsuperscript{12} Ibid., 71.
\textsuperscript{13} Ibid., 26.
geometry relaxes and transforms the law of homogeneity. These differences, however, stem from a deeper disparity concerning the nature of the mathematical objects considered in the two: ancient geometry is concerned with ideal objects having visible images, which have an inherent, noetic nature; in contrast, Cartesian geometry is concerned with algebraic objects that exist in the mind and are figural only insofar as they have been transposed into a visible bodily form, i.e., they do not image ideal (noetic) beings. This, in turn, is connected to a more overarching difference in the nature of the mathematical activity of the mathematician himself. According to Lachterman, in writing the *Elements* Euclid is acting primarily as a teacher who “strives to find means fitted to the choiceworthy end of allowing a student to learn what is intrinsically learnable,” which Lachterman calls “the geometrical version of practical prudence” or “didactic *phronesis*.” In contrast to this, Lachterman understands Cartesian geometry to be governed by the desire to find a creative technique that brings new objects into existence. Thus, while Euclidean geometry is “bent on inculcating the appropriate virtues in the learner *qua* learner, the Cartesian ethos . . . concentrates on exhibiting the virtuosity of the artisan *qua* inventor.”

This general overview of the contrast between Euclidean and Cartesian geometry is the backdrop against which Lachterman’s account of the nature of Descartes’s mathematics must be understood. This diagnosis comes in the form of identifying what Lachterman calls the “epistemic signature” of the *Geometry*, which he explains as the “way or ways certain acts and forms of knowing become incarnate in the contents and results that materially define the work”;

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14 Ibid., 122. Lachterman also refers to this as “mathematical *phronesis*,” and glosses it as “the fitting of appropriate means to ends worthy of choosing, rather than the determination of ends by the accessibility of means” (32). Elsewhere he contrasts “Euclidean prudence,” which shows itself “as responsiveness to nontechnical shapes and the pre-understanding which they evoke,” with “Cartesian prudence,” which “amounts essentially to being loyal to the regulations of technical exactitude” (174).

15 Ibid., 151.
such a signature serves as an indication of the work’s ontological limits in the sense that “only what satisfies the conditions of possibility and intelligibility governing the discursive structure of that work can be promoted to the rank of a geometrical entity in good standing.”

There are four “principal characteristics of Cartesian Geometry” that Lachterman identifies as a part of Descartes’s “implicit ‘metaphysics of geometry.’” They are (1) constructive problem-solving, (2) the homogeneity of mathematical objects, (3) the “principle of kinematic determinateness,” and (4) the use of algebraic analysis. These four characteristics work together and thus must be understood in conjunction, while also being seen in contrast with the ancient geometrical mode.

Regarding the first, whereas ancient geometry was primarily concerned with proving theorems, as seen in the very structure of the Elements as a chain of geometrical truths deduced from definitions and axioms, Cartesian geometry is exclusively concerned with solving problems. This can be seen in the central place held by the Pappus locus-problem in the Geometry, the solution of which was both the “immediate occasion” for this work as well as the starting point for explicating “how to go about solving all geometrical problems.” Moreover, the nature of problem-solving is different in the ancient and Cartesian modes: for Euclid or Apollonius, the solution to a geometrical problem consists in the demonstration of the existence of a geometrical entity with given properties; for Descartes, “to solve a problem is . . . to construct a problem, not, of course, in the sense of making up a problem, but as the successful

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16 Ibid., 143. Lachterman connects his idea of an “epistemic signature” with Descartes’s own phrase “the metaphysics of Geometry” (Letter to Mersenne, 9 January 1639 – AT II 490), although he admits to extending the usage of that phrase beyond what Descartes seems to mean by it.
17 Lachterman, Ethics of Geometry, 191.
18 Ibid., 144. The Pappus problem will be discussed in detail below in § 5 of this chapter.
process of finding and exhibiting the relevant geometrical item(s) satisfying the conditions set out in an algebraic equation.”

While we have not yet examined what is involved in this algebraically-governed exhibition (which we will come to shortly), it suffices for now to highlight that what distinguishes Cartesian problem-solving from its ancient precursors is the emphasis on the construction of such problems according to algebraic expressions. Constructive problem-solving is thus both a principal and distinguishing characteristic of Cartesian geometry.

Second, regarding the heterogeneity or homogeneity of mathematical objects, ancient geometry respected the distinctions between different kinds of quantities, including the distinction between multitude and magnitudes as well as the heterogeneity of the different kinds of magnitudes (lines, figures, solid), along with the attendant limitations on operations and comparisons (ratios and proportions) involving quantities of different kinds. Cartesian geometry, in contrast, seeking uniformly applicable operations, deals with a purely homogeneous mathematical object, under which all the various kinds of quantities are subsumed. It thus does not uphold the heterogeneity (of numbers and kinds of magnitudes) respected by ancient geometry; instead, it reduces all multitudes to a division of magnitudes, while interpreting dimension (e.g., square, cube) as “a benchmark indicating the sequential order and the relative measurement of the terms ingredient in an algebraic equation.”

Central to this understanding of the object of Cartesian geometry is the “arbitrary choice of a line-segment to be the unit-measure,” whereby Descartes “circumvent[s] the issue of incommensurability, the crux of the

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19 Ibid., 191-92.
20 Ibid., 167. Lachterman explicates this last point elsewhere by saying: “Descartes invites us to reinterpret the dimensional differences expressed by exponents (x², x³, xⁿ) as marking positions in a continuous ordering of ratios, not as signifying perceptible (or, in cases where n > 3, imperceptible) distinctions (as, say, between a 2-dimensional square and a 3-dimensional cube)”; ibid., 192.
Euclidean tradition.” The homogeneity of mathematical objects is, then, a second distinguishing hallmark of Cartesian geometry, which is connected to the first insofar as this leveling of mathematical differences is what allows for all mathematical objects to be treated under one constructive, algebraic rubric.

Third, the “principle of kinematic determinateness,” as Lachterman calls it, is Descartes’s criterion governing which curves are admitted into geometry. This criterion concerns the movement of the instrument by which a curve is produced: “If there is only one continuous movement or if several motions succeed one another in such a way that the later motions are ‘completely regulated’ by those that precede them, then and only then are the resulting curves to be called ‘geometrical.’ Otherwise they are termed ‘mechanical.’” While ancient geometry also made a distinction between geometrical and mechanical curves, Descartes redraws the boundary between the two, expanding the range of curves admitted into geometrical consideration. The mistake of the ancients, according to Descartes, was to think that any instrumental origin of a curve compromised its intelligibility, whereas he believes that the strict regulation of the construction of a curve according to determinate ratios guarantees the

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21 Ibid., 192. Incommensurability can be called the “crux of the Euclidean tradition” in that it marks the crucial difference between multitudes and magnitudes, for which reason it is central to the heterogeneity governing the divisions between the ancient mathematical sciences.

22 Ibid., 170. This topic will be discussed in detail in § 5 below. For now it suffices to see that the criterion concerning which curves can be admitted into Descartes’s mathematics is a mechanical one that governs the motion of an instrument producing a curve and that the result of this criterion is that every curve admitted into Descartes’s mathematics is constructed by an exactly regulated motion such that the curve itself is exactly determined.

23 The ancients were generally opposed to defining a curve by means of its mechanical or instrumental production, as this was thought to undermine its noetic character; instead they preferred to define a curve by its intrinsic properties. Curves that could only be produced mechanically therefore tended to be excluded from geometry. In contrast to this, Descartes’s criterion is broader and allows for the inclusion of a wider array of curves precisely because it focuses on the mechanical means of production. Lachterman mentions the cissoid and the conchoids as examples of curves that were excluded from ancient mathematics but allowed by Descartes; see ibid., 170. For a good discussion of Descartes’s disagreement with the ancients regarding the admissibility of curves, see A. G. Molland, “Shifting the Foundations: Descartes’s Transformation of Ancient Geometry,” Historia Mathematica 3 (1976).
intelligibility of that curve, regardless of how that genesis was performed. “[B]y producing a curve in an exactly regulated manner, one can know with certainty the origin and the determinate measure of each of its (infinite) points,” which entails that “the curve as a whole can be expressed by a single equation.” In fact, the construction of such curves is governed by the algebraic equations which express them, as is seen in the fourth principal characteristic of Cartesian geometry, the use of algebraic analysis.

For Descartes, “algebraic analysis does not stand on its own but, rather, is employed as an instrument designed to facilitate and (partially) authenticate geometrical constructions.” Thus, on Lachterman’s account, Descartes does not combine geometry and algebra by making geometry algebraic or algebra geometric, as is often said; rather, he uses algebra as “a tool for calculating the relations among the lengths and distances figuring in a problem-complex.” This algebraic analysis is free from any homogeneity requirements and is employed as a problem-solving technique, both of which make it characteristically different from ancient geometrical analysis, which was primarily used as a prelude to a synthetic demonstration. Moreover, the dependence of algebraic analysis on symbolization also highlights its difference from ancient analysis. According to Lachterman, both the use of algebraic letter signs in equations and the representation of all quantities by means of line-segments shows “the Cartesian resolve to treat ‘magnitudes in general’” (an object which we have seen to be completely foreign to ancient mathematics).

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25 Ibid., 158.
26 Ibid., 193.
27 For a good, concise discussion of ancient geometrical analysis, see Klein, *GMTOA*, 154-56.
means of letter signs and that by means of line segments—are closely intertwined in that the symbolic line segments of Descartes’s mathematics are simply another means of representing the general relationships captured in his algebraic equations. As Lachterman says, “the line-segments in Book 1 of the *Geometry* serve only to record in the simplest or most economical way those relations among magnitudes which are discerned or produced via the manipulations of algebraic letters and numerical coefficients.”

Here we come to the heart of Lachterman’s analysis of Descartes’s mathematical procedure, insofar as this “recording” of the relations delineated in an algebraic equation by means of a visual depiction with line-segments is central not only to Descartes’s use of algebraic analysis but to the *Geometry* as a whole. For, on Lachterman’s account, Descartes’s “general method in the *Geometry* is aimed at the exhibition of successful constructions (or constructional procedures) based upon the formal results of algebraic analysis.” In the course of this procedure, a set of abstract relations considered in the mind are encoded into an equation and then embodied in a geometrical construction:

The complete mathematical process of which a construction is the terminus consists of a double transcription. The algebraic equations expressing the ratios of the line-lengths in a given problem transcribe into symbolic notation the ordered sequence of the steps the mind takes in arranging the terms of that problem according to knowns and unknowns . . . . The algebraized equation, in turn, not only encodes and retains that intellectual sequence; it is also the expression of a quantity whose roots are the actual line-segments to be drawn from points at specified distances from the chosen axes. The set of points determined by these measures is the locus of a curve. The transcription or inscription of

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29 Ibid., 168. Lachterman here further explicates the symbolic representation by means of line segments as follows: “these line-segments have ceased to be geometrical, that is, linear magnitudes in their own right or ‘by nature’; instead, they symbolize or explicate the terms in a sequence of ratios among any magnitudes whatever. And it is only with respect to this application of straight line-segments that we can talk of ‘symbolic or abstract magnitude in general’ as the subject-matter of Cartesian *mathesis.*” This topic will be taken up in much greater detail in the following section, wherein we will take an in-depth look at the opening pages of the *Geometry.*

30 Ibid., 156.
the abstract equation into a visible geometrical configuration is the construction being sought.\textsuperscript{31}

The movement here is from the realm of the intellect to the realm of perception, and the direction of this movement “informs us that the marks or shapes accessible to perception are caused to be there by the intellect imaging its own activities.”\textsuperscript{32}

This points to the interaction of the intellect and imagination that is at work in the \textit{Geometry}. Algebraic analysis begins with “symbolic abstraction,” which entails a particular relationship between these two faculties (as discussed at length in the previous chapter), but this is “only a preparatory first step”; “the real work of [Cartesian] geometry” lies in taking the equations generated by means of algebraic analysis “as instructions for the graphical construction of the relevant loci and roots” (i.e., the construction of curves and the lines which determine the points on those curves), a process in which “the intellect and the imagination must somehow be equal partners” in that both are essentially required to complete the task.\textsuperscript{33} According to Lachterman, then, the symbolic abstraction that underlies algebra is, as Klein has shown, a product of the peculiar interaction of intellect and imagination, but this is only the beginning of their shared work; they must then work together to embody and externalize the relations articulated in the algebraic equations by constructing a representative depiction of those relations by means of line-segments.

We are now in a position to see the connection Lachterman draws between the mathematics of the \textit{Geometry} and the account of the mind in the \textit{Rules}. This begins with the unity of the sciences proclaimed in Rule 1: insofar as the sciences can be taken together as a

\textsuperscript{31} Ibid., 195.
\textsuperscript{32} Ibid., 196.
\textsuperscript{33} Ibid., 186.
whole due to the singularity of the source of knowledge—namely, the human mind—those sciences have one common object that is dictated by what is knowable by that mind. Beginning in Rule 4 and explicated throughout the rest of the Rules (as seen in the previous chapter), Descartes restricts this intelligibility to what partakes of order and measure. As a consequence of this, “the beingness of objects, what they are *qua* beings, coincides with their mathematical intelligibility, and this means, their comparability—that is, in the strictest of senses—their ability to enter into determinate ratios and proportions with other, homogeneous beings.”\(^{34}\) From this “homogeneity imposed by ‘human science’ on the totality of what can be known” it follows that everything that is knowable is comparable in the sense that “we can always speak in a determinate manner of one item’s being more than, less than, or equal to, another item.”\(^{35}\) The domain of the knowable is therefore equivalent with the domain of (Cartesian) magnitude, meaning that everything that is knowable is susceptible to mathematical manipulation. In Rule 14, when Descartes prescribes that we deal only with magnitude in general, this mathematical manipulation of the knowable is restricted to that preformed in the symbolism of algebraic equations and Cartesian geometric constructions. Moreover, measure is subordinated to order in that Rule 14 reduces measurement to “the inspection of an ordered sequence of multiples of a unit,” whereby Descartes’s emphasis falls on “preserving exactness in the sequence of steps or moves the intellect makes as it sets about disposing and comparing the terms ingredient in a given problem.”\(^{36}\)

\(^{34}\) Ibid., 178.  
\(^{35}\) Ibid., 179.  
\(^{36}\) Ibid., 181.
Lachterman thus highlights that the beginning of Descartes’s mathematical cognition is the well-ordered progression of the pure intellect, which must then be translated or transcribed into the domain of the sensible via the imagination. As he says:

the imagination has to serve as both the instrument by which and the medium in which the prescribed courses of technical genesis can be both carried out and appreciated for what they produce. This collaboration, in turn, vitally depends on having the intellect place the imagination in its service, not vice versa! That is, images (such as algebraic symbols) must be formed as the records of intellectual motions, not as the by-products of untutored or presystematic perception.  

Mathematical symbols thus capture the very motion of the mind as it moves through its activities of comparisons and deductions. As we saw above, this capturing begins with the production of an equation, which “encode[s] those acts of the mind by which order and the possibility of measure are self-reflectively grasped,” and then proceeds with the construction of a geometrical depiction that embodies the relations delineated in the algebraic equation. Throughout this process, the intellect and imagination must work together to produce symbolic representations in a way that Descartes began to work out in the second half of the Rules, but only brought to complete fruition in the mathematical procedure of the Geometry.

It is thus the mechanical motions of the mind (i.e., the regulated and well-ordered steps taken by the pure intellect) that ultimately ground the symbolic formations of Cartesian mathematics. This is what was foreshadowed earlier when Lachterman said Descartes’s mechanization of mind works together with symbolization in his mathematical procedure. In fact, to spell this out further, we can now say that Lachterman connects the account of mathematical cognition in the Rules with the mathematics of the Geometry by showing that

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37 Ibid., 181-82.
38 Ibid., 186.
Descartes’s mechanistic view of mind underlies the construction that links the symbolism of algebra with the symbolic figures of his geometry. He can therefore conclude that “Cartesian ‘dualism’ is already at issue in mathematics” in that, “[f]or the mind to externalize itself through its productive expression in equations and then in constructions accessible to perception, it must be both independent of, and connected with, the corporeal.”39

This conclusion is in line with the analysis of the previous chapter, while also adding an additional layer to the account. Insofar as the operationally or methodologically dualistic mind-world relationship contained in the Rules is built into the mathematical cognition of Rule 14, it was concluded that that dualism was also built into the symbolic concept of number which that cognition makes possible. With the help of Lachterman, we have now begun to see how that same dualism is at work in the Geometry—namely, in the double transcription of abstract relationships, first into algebraic equations and then into geometric constructions—whereby we have our first indication that the mind-world relationship of the Rules is built into the mathematics of the Geometry. This, however, will be spelled out in greater detail in our own ensuing analysis of the symbolic mathematics employed in that work.

Before proceeding with that analysis, however, let us conclude this section by examining the brief comments Lachterman makes about space, as this will set up the analysis of Descartes’s concept of space later in this chapter. It should be noted first that Lachterman cautions that “[t]he locale of Greek geometry may be foreign to the modern conceptions of extension and space,” whereby he echoes Klein’s claim (mentioned above in §9 of Chapter 1) that “Euclidean space”—at least as it is commonly understood—is not in fact the domain of ancient geometrical

39 Ibid., 202.
objects.\textsuperscript{40} In seeking Descartes’s concept of space in the \textit{Geometry}, we must therefore resist the temptation to assume that there is some clear and univocal understanding of space that governs all geometrical works. Instead, we must examine the concept of space as it emerges from the proceedings of this particular work itself.

As we have just seen, Cartesian geometry depends upon an externalization of abstract relations by means of symbolic representation. Lachterman describes this movement as a “transposition of mathematical intelligibility and certainty . . . from the interior forum of the mind to the external forum of space and body.”\textsuperscript{41} In this process, what is contained in the mind is “bodied forth in a forum not of the mind’s own making, however much it stands under the authority of mind.”\textsuperscript{42} This forum is the spatial domain of Cartesian geometry, which is not of the mind’s own making in that it is other than the mind and exists separately from it, while also being under the authority of the mind in that the activities of the mind dictate what can be embodied and exhibited in it. The domain of Descartes’s geometry can thus be described as “the ‘imaginative,’ but not merely imagined, space of geometrical construction.”\textsuperscript{43}

Lachterman emphasizes that there are limitations to what is admissible into the space of Cartesian geometry and that these limitations stem from the mind’s constructive activity: “The limits to the power of intellect to image itself in the domain of perception are given by the very restriction on admissibility (that is, the requirement of kinematic determinateness) that

\textsuperscript{40} Ibid., 28. Elsewhere, Lachterman points out that “there is no term corresponding to or translatable as ‘space’ in [the] generalized sense anywhere to be found in Euclid’s \textit{Elements}”; ibid., 80. As mentioned in Footnote 100 of Chapter 1, the question of space in the \textit{Elements} cannot be pursued here, as it would require a detailed study of that work. Suffice it to say that the establishment of the symbolic character of Cartesian space found later in this chapter entails that this space cannot be the domain of Euclid’s non-symbolic geometry.

\textsuperscript{41} Ibid., viii.

\textsuperscript{42} Ibid., 197.

\textsuperscript{43} Ibid., 186.
simultaneously ensures the exactness and certainty of constructions.”\(^{44}\) In other words, the
criterion that determines what can be constructed by the mind also determines the limits of
Cartesian space. This is evidenced by the fact that “the range of [geometrical] intelligibility . . .
does not coincide with the domains either of perceptual accessibility or of algebraic
tractability.”\(^{45}\)

One important example of the limitations to what is admissible into Descartes’s geometry
is to be found in his acknowledgment of the existence of imaginary roots but his rejection of
them as possible solutions to a problem.\(^{46}\) As Lachterman says,

> Not every equation can be “constructed” in accordance with Cartesian regulations, and
this means that not every intellectual motion or sequence of motions can in fact find its
corporeal representative or surrogate. Analogously, any rightly formed equation has as
many roots as exponents of its unknown terms, but some of these roots will be
‘imaginary,’ that is, strictly speaking, \textit{unimageable} and hence without potential purchase
on the extended world.\(^{47}\)

Thus only positive and negative roots are admissible into Descartes’s geometry “since there is no
‘space’ in the local expanse determined by the principal lines in which ‘imaginary’ roots can be
inscribed.”\(^{48}\) The space of Cartesian geometry is therefore determined by what can be
constructed within it, which is delineated by the “principal lines” that determine and ensure the
intelligibility of everything constructed in relation to them. This important role of the “principal

\(^{44}\) Ibid., 196.
\(^{45}\) Ibid. Lachterman explains this point further as follows: “\textit{Only} those equations all of whose exponents are rational
correspond to admissible curves (those Descartes calls ‘geometric’ and Leibniz later calls ‘algebraic’ in distinction
from ‘transcendental’ curves). Conversely, only those curves constructed in the requisite manner succeed in
transcribing algebraic equations of the appropriate form. This means that there are curves occurring in the sensible
domain (the logarithmic spiral, say) \textit{or} producible by instruments (the quadratrix, say) as well as equations
expressing authentic problems (such as \(x^2 + x = 1\), the formula for cutting an angle in a given ratio), that
permanently elude the grip of Descartes’ technique, despite his being thoroughly acquainted with most of these
formations.”
\(^{46}\) See Descartes, \textit{Geometry}, 453-54.
\(^{47}\) Lachterman, \textit{Ethics of Geometry}, 182.
\(^{48}\) Ibid., 157.
lines” (as Descartes’s calls them) will be discussed in depth below in § 5, but for now its suffices to point out that these principal lines later develop into the coordinate axes of the Cartesian plane.\footnote{For an overview of this historical development, see Carl B. Boyer, \textit{History of Analytic Geometry} (Mineola, NY: Dover Publications, Inc., 2004), esp. Chs. VI – VIII.} The domain of Descartes’s geometrical space is thus limited to what can be handled by his symbolic mathematical procedure, and this domain is defined and delimited by his principal lines.

Here we catch our first glimpse of the symbolic nature of Cartesian space: insofar as the only objects that exist within its confines are the symbolic constructions of Cartesian geometry, this space marks out a symbolic domain. Yet this is only a first step toward seeing that Descartes’s concept of space is itself a symbolic construction in that the domain marked out by that conception is determined and articulated by the symbolic construction of the principal lines. The argument for this last point, however, must be postponed until the end of our analysis of the \textit{Geometry}.

\section*{§ 3. The Symbolic Mathematics of the \textit{Geometry}}

The distinctive characteristics of the \textit{Geometry} are apparent from its opening page. In the First Book, entitled “Of problems that can be constructed without employing anything other than circles and straight lines,” Descartes begins by declaring that “All the problems of geometry can easily be reduced to such terms that it is only necessary thereafter to know the length of certain straight lines in order to construct them” (369). With this, he sets up his program of constructive problem solving using line lengths. He then goes on, drawing an analogy with arithmetic, to
explain that these line lengths can be found by means of the simple operations of addition, subtraction, multiplication, division and the extraction of roots (which he calls a species of division).

The addition and subtraction of lines to generate other longer or shorter lines calls for no explication, as this is a standard practice. The multiplication and division of lines resulting in another line, however, is revolutionary, so Descartes must explain how to do this. In this explanation, he redefines these operations for his new geometry. In pre-Cartesian geometry, the only way to multiply one line by another is to construct a figure composed of those two lines set at right angles with each other. In this way, the multiplication of two geometric objects of the same genus (namely, two lines) results in a new geometric object of a different genus (namely, a rectangle). Descartes’s new understanding of multiplication wipes out this change in genus or dimensionality, whereby we see the first indication of the homogenization of mathematical objects.

Descartes’s new understanding of multiplication amounts to finding a fourth proportional. He describes it thus: “taking one [line], which I shall call unity [unité] in order to relate it as closely as possible to numbers, and which can ordinarily be chosen arbitrarily [a discretion], and then taking two others, find a fourth which is to one of the two as the other is to unity—which is the same as multiplication” (369-70). Before considering his example of this procedure, let us first briefly note some of the other novel aspects of Descartes’s geometry that are being introduced here, namely, the arbitrary selection of a particular line as the unit in relation to which the lengths of all other lines are measured, and the collapsing of the distinction

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50 Descartes advocated this approach himself in Rule 18, but he excludes it from the Geometry. This serves as an initial example of the development his mathematical thought had undergone since the composition of the Rules.
between multitude and magnitude (and therewith the distinction between arithmetic and geometry). Moreover, Descartes tacitly admits that he is redefining multiplication when he says that finding the fourth proportional is “the same” as multiplication, rather than that it is multiplication.

Descartes goes on to redefine division as the inverse of this understanding of multiplication and the extraction of roots as the finding of mean proportionals, after which he gives examples to explicate these newly understood operations. It is here that we will begin to see the mathematical cognition of Rule 14 being employed in the *Geometry*. His example of multiplication is as follows: “let AB be unity [see Figure 1], and let it be necessary to multiply BD by BC; I have only to join the points A and C, then to draw DE parallel to CA, and BE is the product of this multiplication” (360).

What are we to make of this diagram? Understood in a Euclidean fashion, this is a perfectly good diagram that exhibits two similar triangles with all the attendant properties, including proportionality between the lines that are their sides. But what does this have to do with multiplication? As a Euclidean diagram, there is nothing to indicate that any calculative

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51 The proof that BE is the fourth proportional (i.e., that $AB : BD :: BC : BE$) lies in the similarity of the triangles ACB and DEB—see Euclid *Elements* Book VI, Proposition 2.
operation has taken place. How, then, are we to understand this diagram as exhibiting multiplication, and why does the multiplication of the lines BD and BC result in another line, namely, BE?

It is only according to the new definition of multiplication that Descartes has just given that we can understand this diagram to exhibit multiplication. The diagram itself, however, gives no indication that this is how it is to be understood, as evidenced by the fact that Descartes uses the same diagram to exhibit the inverse operation of division. The interpretation of the diagram as exhibiting multiplication must therefore be read into the diagram, which is exactly what Descartes asks us to do when he gives a description that allows us to interpret the diagram in a new way that meets his new definition. Even as a Cartesian diagram, then, it does not exhibit multiplication on its own, although we can understand it to exhibit that operation insofar as we understand that operation to consist in the finding of the fourth proportional.

Beyond merely exhibiting the operation, the diagram also enables us to effect this new operation by actually allowing us to find the fourth proportional of the two lines being multiplied. To do this, however, it is absolutely necessary to begin by designating some line as unity, otherwise there would be no way to find a fourth proportional because there would only be two lines in play. This shows just how crucial the designation of unity is; without it there would be no way to perform the operation of multiplication. The multiplication of line BD by BC is meaningless without reference to some line designated as unity. Yet, as Descartes himself has

52 To exhibit division using Figure 1, Descartes simply says “If it is necessary to divide BE by BD, having joined the points E and D, I draw AC parallel to DE, and BC is the result of this division” (370). In addition to the explication of multiplication and division, Descartes also here exhibits the extraction of a square root as the finding of a mean proportional, for which he uses a different diagram, but the present consideration will be confined to the example of multiplication, as the general procedure is the same in all three procedures.
said, that line can be chosen arbitrarily, which indicates that the multiplication of two lines by each other has no determinate result in itself (unlike the multiplication of two numbers). The multiplication of line BD by line BC only results in a line of length BE when the unit is stipulated as line AB. If some other line had been chosen as the unit, e.g., shorter than AB, then BE would also be different, e.g., longer, while BD and BC—the two lines being multiplied—would stay the same. In fact, the multiplication of these two lines could result in a line of any length; all one has to do is vary the unit line. This provides for a striking contrast with the traditional understanding of multiplication, where the product of two factors is only ever of one determinate size.

The specificity of the lines and lengths involved in this diagram are therefore all interdependent. None of them can be separated out and considered by themselves; they must be considered only in relation to each other, for if a different line were chosen as unity the rest of the diagram would have to be redrawn. The diagram must therefore be understood as a whole in order to represent multiplication. This, coupled with the arbitrariness of the unit and the resulting arbitrariness of the product (suspending the question of incommensurability), reveals that the actual line lengths making up the diagram are not all that is important in Figure 1. While the diagram does allow us to effect the operation of multiplication by finding the determinate length (BE in units of AB) that is sought, it can also be understood as an articulation of the general relationship that holds between the quantities at play in Descartes’s understanding of multiplication, namely, as unity is to the first factor, so is the second factor to the product.\footnote{Or, in standard notation, unity : the first factor :: the second factor : the product. Henry Higuera, in a helpful lecture on various historical understandings of multiplication, points out that Descartes has here redefined the unit “as the ‘multiplicative identity ‘1,’” such that for all $a$, $1 \cdot a = a$; Henry Higuera, “Multiplication, Ancient and
this understanding, the lines making up the diagram have no real meaning in terms of their determinate lengths, but rather are used to represent or visually depict the ratio that holds between the various terms that make up Cartesian multiplication.

This diagram can therefore be read in two ways: it can be taken as a (Euclidean) depiction of triangles made up of determinate lines, which can then be interpreted as and used to effect Cartesian multiplication, and, finally, it can be taken as a representation of the general relationship that defines Cartesian multiplication, a representation which, although made up of determinate lengths (based on the choice of AB), has no determinate meaning in itself. This diagram, then, has both a non-symbolic and a symbolic interpretation; in the former, the diagram is made up of lines of determinate length which are the object of a mathematical operation, while in the latter those determinate lines are merely used to represent a general relationship between indeterminate magnitudes.54

It is in this latter understanding that we begin to see the employment of the mathematical cognition of Rule 14 in the mechanics of the *Geometry*. We saw in the previous chapter that that cognition entails a combination of a general, indeterminate abstraction existing in the intellect with a determinate, bodily representation that exists in the imagination. In the case at hand, the intellect is concerned with the general relationship that defines multiplication, namely, the proportion between four indeterminate magnitudes—an arbitrary unit-line, two given factors, and a product-line of whatever length is determined by the previous three terms. The imagination

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54 While this last point will be fleshed out in more detail below, let it be recalled here that this combination of indeterminate object and determinate representation is the hallmark of a symbol in Klein’s sense (as discussed in §§ 6-7 of Chapter 1 and § 8 of Chapter 2).
then provides a bodily depiction of those terms that exhibits that general relationship, which is externalized and presented to the eyes in Figure 1. While this depiction is made up of lines of determinate lengths, those determinate lengths are completely incidental to the diagram’s function as a depiction of the general relationship. On this interpretation of the diagram, then, individual lines merely serve to represent indeterminate quantities in a general relationship.

Here we have a furthering of the symbolic construction begun in Rule 14. In § 8 of Chapter 2, it was concluded that Rule 14 lays out the epistemological basis that underlies the process of symbolization but that the complete process of symbolization only occurs when the indeterminate object of the intellect enters into the five actual operations of mathematical practice (addition, subtractions, multiplication, division, taking roots) via its bodily representation, for it is only at that point that the indeterminate object is actually treated determinately (rather than merely being signaled by some determinate representation). This is exactly what happens as a result of the coexistence of the two interpretations of Figure 1. In using the Euclidean understanding of the diagram the lines are of determinate length and can therefore be considered and manipulated determinately, while on the new Cartesian understanding those determinate lengths actually stand for indeterminate magnitudes. The lines of this diagram are therefore determinate in themselves while also serving as the representations of indeterminate magnitudes. Their use and manipulation therefore puts the indeterminate content for which they stand into mathematical operations in a way that would otherwise be impossible because indeterminate magnitudes cannot enter into mathematical calculation on their own, as they have no quantitative value according to which they can be manipulated. When, however, indeterminate magnitudes are represented by determinate magnitudes, which do have
an inherent quantitative value and therefore can be manipulated, the indeterminate magnitudes can enter into the course of mathematical operation, albeit by proxy. It is significant here, though, that while the two interpretations of Figure 1 function together, they are conceptually separable, which shows that the distinction between the different orders they embody has not been completely collapsed. With the introduction of algebraic letter signs, however, which immediately follows in the *Geometry*, this is no longer the case.

Upon completing the explication of how to effect basic arithmetical operations by means of line segments, Descartes introduces algebraic letter signs into his mathematical procedure, thereby taking a further step in symbolization. He does this by saying the following:

But often one has no need to trace these lines on paper in this way, and it suffices to designate them by certain letters, each [line] by one [letter]. Thus, in order to add the line BD to GH, I name the one $a$ and the other $b$, and write $a + b$; and $a - b$ in order to subtract $b$ from $a$; and $ab$ in order to multiply the one by the other; and $\frac{a}{b}$ in order to divide $a$ by $b$; and $aa$ or $a^2$ to multiply $a$ by itself; and $a^3$ in order to multiply yet again by $a$, and so on to infinity. (371)

Descartes goes on to emphasize the homogenized dimensionality of the referent of his algebraic letter signs, saying that “by $a^2$ or $b^3$ or the like, I ordinarily mean only simply lines, although in order to make use of the names used in Algebra, I call them squares, or cubes, etc.” (371). He then explains how to operate with letter signs of different degrees in such a way that avoids any issues of heterogeneity. Finally, he advocates making a list of the letter signs used to name the individual lines so as not to forget them.

The algebraic letter signs of the *Geometry* thus stand for individual line segments, and it is easily discernable that they must correspond to the interpretation of Figure 1 in which the line-segments themselves stand for indeterminate magnitudes. Note, first, that these letter signs are
not determinate geometrical magnitudes and thus, as drawn marks on the page, cannot be taken in the manner of the Euclidean interpretation of Figure 1. They do, however, enter into the course of calculation, but Descartes has to give directions for how to operate with them because there is no inherent way to calculate with these letter signs which have no quantitative value in themselves. In this way, then, Descartes’s letter signs \((a, b, z, x)\) are inherently geared toward capturing indeterminacy (they have no definite numerical values) and allowing it to enter into mathematical calculation (they can be added, subtracted, multiplied, divided). Moreover, Descartes explicitly says that the use of these letter signs replaces the need to draw line segments on the page, whereby he shifts the focus even further away from determinate magnitudes, indicating that the algebraic letter signs need not refer to any determinately drawn line-lengths. The practice of using algebraic letter signs is therefore just another way of creating an external representation of what is considered internally. Thus, these letter signs serve simply to re-express what was captured in the new Cartesian interpretation of Figure 1. It is clear, then, that the expression by line-lengths and the expression by algebraic letter signs are simply two different ways of articulating the same thing, namely, indeterminate magnitudes and their general relationships.\(^{55}\)

While both of these depictions are symbolic in that they are determinate representations of indeterminate magnitudes that are used to effect mathematical calculation, they differ in that the symbolic line-segments are geometrical figures that have determinate magnitudinous quantity while standing for an indeterminate quantity, whereas the algebraic letter signs simply have no

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\(^{55}\) In this way, the transition to algebraic letter signs retrospectively provides further evidence for a symbolic interpretation of Figure 1 in that it shows that generic letter signs with no determinate geometrical value serve just as well to represent the terms of the multiplicative ratio as lines of determinate length.
such determinate value. As drawn marks on a page, these letter signs are determinate bodily figures, but insofar as they are not geometrical figures they do not have any inherent quantitative value that can be used in mathematical calculation. Thus there is no second interpretation of how to read these algebraic letter signs; their use in mathematical operations unavoidably entails the full conflation of general indeterminate content and determinate representation.

While Descartes’s algebraic letter signs might therefore be considered more thoroughly symbolic than his depictions by line-lengths, insofar as those letter-signs cannot solve geometrical problems without reference to some geometrical construction, it is the duality of Descartes’s geometrical figures that are essential in characterizing his new mathematical science as a symbolic geometry; for it is by giving his figural depictions an algebraic character (i.e., the same conceptual structure as an algebraic symbol) that Descartes brings the symbolism of algebra into the realm of geometry.  

Thus it is the symbolic character of Descartes’s figures that allows him to connect his algebraic letter signs with his geometrical constructions.

The mathematical procedure of the *Geometry* rests on this ability to move back and forth between the algebraic and geometrical expressions of Descartes’s symbolic mathematical objects. This begins to become clear in the description Descartes gives of his general analytic procedure for solving problems, which immediately follows his discussion of the use of algebraic letters signs. To solve a given problem, he says, one should first assume the problem to be

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56 As Klein says, “Descartes does not, as is often thoughtlessly said, identify ‘arithmetic’ and ‘geometry’—rather he identifies ‘algebra’ understood as symbolic logistic with geometry interpreted by him for the first time as symbolic science”; Klein, *GMTOA*, 206.

57 This highlights just how important but potentially misleading Descartes’s figural depictions are: they define the *Geometry* as a symbolic mathematical science while also giving it the appearance of traditional (i.e., non-symbolic) mathematics. These figures accomplish this by preying on the imagination, insofar as we unavoidably see the line lengths as having a determinate magnitude, while they in fact represent something quite different. Thus, while they appear as traditional geometrical magnitudes, they are in fact an entirely new, symbolic, mathematical object.
already solved and then give names to all the lines that are needed to construct it, including both the known and the unknown lines. Then, “without considering any difference between known and unknown lines, one should go through the difficulty according to the order which shows most naturally the mutual dependence between these lines, until one has found a means of expressing a single quantity in two ways, which is called an equation because the terms of one of these two ways are equal to those of the other” (372). It is then necessary to find as many equations as there are unknown terms; if it is not possible to do this, the problem has not been completely determined, although it can be made completely determined by simply assigning known values for the unknowns. Once all the equations have been found, they must be considered in order, either by themselves or in comparison with others, in order to construe (or explain—*expliquer*) each of the unknown lines (372). Then, by putting these equations in relation to each other and “untangling” (*demelant*) them, they can be reduced to the point that “there remains only one [unknown line] equal to some others that are known” (373).  

While we will not see the full employment of this method until later when we consider the more complicated example of the Pappus problem, we can begin to see Descartes’s combination of geometry and algebra in the simple examples of solving quadratic equations offered in Book 1. In order to find the root (i.e., the unknown line $z$) of the equation $z^2 = az + bb$, Descartes says:

I make the right triangle NLM [see Figure 2], whose side LM is equal to $b$, the square root of the known quantity $bb$, and the other LN is $\frac{1}{2} a$, the half of the other known quantity, which was multiplied by $z$, which I suppose to be the unknown line. Then,

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58 The final remaining unknown can also be a higher power of the unknown, leaving a higher order equation, but this makes no significant difference to the general procedure.
extending MN, the base of the triangle,\textsuperscript{59} to O, so that NO is equal to NL, the whole line OM is \( z \), the line sought. And it is expressed in this way: 
\[ z = \frac{1}{2} a + \sqrt{\frac{1}{4} a^2 + b b} \] . (375)

Figure 2

To understand what is going on here we must move back and forth between the algebraic expressions and the geometrical diagram. The key feature of the diagram is that as a Euclidean figure \( LM^2 = OM \cdot PM \).\textsuperscript{60} The equivalent relationship can be found in the algebraic equation by manipulating it as follows:

\[
\begin{align*}
z^2 &= az + bb \\
z^2 - az &= bb \\
z(z - a) &= bb
\end{align*}
\]

The parallelism between this equation and the diagram can be established as follows. Note, first, that \( PM = OM - OP \); then, taking OM as \( z \), and noting that OP = \( a \) (because NL is \( \frac{1}{2} a \)), it becomes clear that \( PM = z - a \). Therefore, \( OM \cdot PM = z(z - a) \), all of which is equal to \( LM^2 \), which by assumption is equal to \( b^2 \). In this way, we can see that the diagram has been constructed to capture the relationship expressed in the algebraic equation. Once this parallelism

\textsuperscript{59} In modern parlance, this would be called the hypotenuse, but as Olscamp points out in his translation, the hypotenuse “was commonly called the base at that time”; Olscamp, trans., Discourse on Method, 181 n. 1.

\textsuperscript{60} Or, in more properly Euclidean language, the square on LM is equal to the rectangle made up of sides OM and PM (see Euclid Elements Book III, Proposition 36). It is noteworthy that Descartes here implicitly takes the multiplication of two lines to make a rectangle as equivalent to multiplying them algebraically.
has been established, the value of $z$ can be found by noting that $OM = ON + NM$, and that according to the Pythagorean theorem $NM = \sqrt{NL^2 + LM^2}$; then, plugging in the algebraic letter signs for the line lengths, $OM = ON + NM$ becomes $z = \frac{1}{2} a + \sqrt{\frac{1}{4}a^2 + b^2}$. Given that $a$ and $b$ were assumed to be known, the value of the unknown $z$ can then be found using this equation.

Descartes goes on to solve two other forms of quadratic equation (one using the same diagram, while the other requires a different construction), but for our purposes the above serves as a prime example of the back-and-forth movement between algebraic expression and geometrical construction. Descartes begins with an algebraic equation that captures a general relationship between indeterminate magnitudes, namely, that the square of some unknown magnitude is equal to the sum of its root multiplied by some known magnitude and some other known square magnitude. He then constructs a geometrical figure that depicts that relationship with line-lengths, associating the algebraic letter signs from the equation with various parts of the diagram. In doing so, it is clear that the line-lengths stand for the same indeterminate magnitudes that the algebraic letters stand for. This indicates that the diagram functions symbolically, but the line-lengths are also used non-symbolically in the appeal to their geometrical arrangement both when establishing the parallelism between the geometrical diagram and the algebraic equation and when using the diagram to re-express the value of the unknown in algebraic terms.

At no point in this procedure is there any determinate magnitude being considered.\footnote{Note that Descartes does not even designate a line-length as the unit in this situation since there is no need to do so.} Instead, Descartes is only ever concerned with a general relationship between indeterminate
magnitudes, which have been represented in two different ways on the page, namely, by written
letter signs and drawn figures. It is this combination that allows Descartes to use the geometrical
diagram to find an algebraic expression for the unknown line that is sought. In the process, both
representations enter into mathematical calculation and are manipulated in a determinate
manner—the letter signs when they are moved around in the equation, and the figure when it is
considered in a traditional geometrical fashion. There is, therefore, a twofold symbolism at work
here, an algebraic one and a geometrical one; in both, indeterminate magnitudes are considered
and manipulated determinately in the course of mathematical operations. This, then, is the
symbolic mathematics of Descartes’s *Geometry*.

§ 4. The Connection between the *Rules* and the *Geometry*

The previous section’s discussion of Descartes’s symbolic mathematics allows us to
highlight the key connections between the *Rules* and the *Geometry*. Most importantly, it can
now be concluded that, insofar as the mathematical cognition of Rule 14 is employed in the
mathematical procedure of the *Geometry*, the epistemological underpinnings of that cognition are
also present in that work. Thus, insofar as it was concluded (in § 8 of Chapter 2) that the
methodological or operational dualism found in the *Rules* was built into the symbolic concept of
number made possible by the account of Rule 14, it can now also be concluded that that
understanding of the mind-world relationship is built into the *Geometry*; for the mathematical
procedures of the *Geometry* have now been shown to complete the process of symbolization that
was made possible by the epistemological considerations of the *Rules*. 
Moreover, it can now also be seen that the *Geometry* represents a coming to full fruition of the mathematics that Descartes began to sketch out in Rules 15-21, as a brief review confirms. In Rule 15 Descartes advocated externalizing the bodily representations of the imagination and presenting them to the senses by drawing figures; while the figural depictions used in the *Geometry* differ from those presented in Rule 15, it is clear that they serve the same purpose. In Rule 16 Descartes extended the symbolic construction of Rule 14 through the use of algebraic practices by advocating the replacement of the figural representations of Rule 15 with algebraic letter signs, which are then employed to consider a problem in the most general possible terms; while the details concerning how algebraic letter signs are used in the *Geometry* are also slightly different, there is still a clear parallel in the algebraic procedure outlined in Rule 16 and the procedure Descartes actually employs in the *Geometry*. In Rule 17, Descartes said that knowns and unknowns must be treated in the same way; this was repeated as a part of the general analytic procedure for solving problems outlined in the *Geometry*. In Rule 18, Descartes claimed that only the four basic arithmetical operations are needed to find an unknown magnitude, and he then laid out a way of performing these operations on the page; the same is said and done at the outset of the *Geometry*, although the operations are performed quite differently and to better effect, displaying a significant advancement over what Descartes began to work out in Rule 18.62 Finally, in Rules 19-21 (which exist only as enunciations), Descartes seems to give general

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62 While Rule 18 does advocate the use of unit lines, it lacks the *Geometry*'s arbitrary imposition of unit lines of any length. Moreover, the multiplication of a line by line resulting in a line is completely absent from Rule 18.
guidelines for the manipulation of equations; this, too, is central to the *Geometry*, although we have only seen preliminary indications of this.  

This brief survey provides further evidence of the close connection between the account of mathematical cognition in Rule 14 and Descartes’s mature mathematical practices in the *Geometry*. Just as the mathematical techniques found in Rules 15-21 were an attempt to put the mathematical cognition of Rule 14 into practice, so too are the more developed versions of those techniques found in the *Geometry*.

Here, then, we have established the pertinent connection between the account of mind in the *Rules* and the mathematics of the *Geometry*, which puts us in a position to take up our consideration of Descartes’s mathematical concept of space. This consideration is initiated in the following section, although it is not brought to completion until the end of the present chapter.

§ 5. The Pappus Problem, Descartes’s Principal Lines, and the Admissibility of Curves

At the end of Book 1 of the *Geometry*, Descartes gives a general introduction to his solution of the Pappus problem. This problem from ancient geometry served as something of the initial impulse for the writing of this work, and Descartes’s solution is intended as a demonstration of the power and superiority of his mathematical method. In presenting his general solution to this problem, Descartes introduces his use of “principal lines” (*les principales [linges]—383), which (we will see) are intimately connected with his criterion for the

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63 Descartes provides an extended discussion of the manipulation of equations in Book III of the *Geometry*, but a consideration of this topic is beyond the scope of the present investigation.

64 For an overview of the relevant history, see Sasaki, *Descartes’s Mathematical Thought*, 205-25.
admissibility of curves. As these issues serve as our gateway to the concept of space at work in
the *Geometry*, we must familiarize ourselves with the former as a propaedeutic to our discussion
of the latter. In doing so, we will confine our focus to that which is relevant to the present
investigation, so as not to be led astray by the many other questions presented by these portions
of the text. ⁶⁵

We begin with Descartes’s introductory discussion of the Pappus problem in Book 1.

After quoting Pappus’s presentation of the problem in Latin, Descartes’s restates the problem in
his own words as follows:

Having three, or four, or a greater number of straight lines given in position, a point is
required from which one can draw as many other straight lines, one on each of the given
lines, which makes a given angle with them; and the rectangle contained by two of those
lines which have been drawn from the same point has the given proportion with the
square of the third line, if there are only three lines, or else with the rectangle of the other
two, if there are four. Or again, if there are five, the parallelepiped composed of three has
the given proportion with the parallelepiped composed of the two which remain and
another given line. Or, if there are six, the parallelepiped composed of three has the
given proportion with the parallelepiped of the other three. Or, if there are seven, that
which is produced when four are multiplied by each other has the given ratio with that
which is produced by the multiplication of the other three and, again, another given line.
Or, if there are eight, the product of the multiplication of four has the given proportion
with the product of the other four. And thus this question can be extended to all other
number of lines. Then, since there is always an infinity of different points which can
satisfy what is demanded here, it is also required to know and to trace the line in which
they are to be found. ⁶⁶ (379-80)

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⁶⁵ For a helpful and thorough discussion of the various matters discussed in this section (as well as those that are
not), see Henk J. M. Bos, *Redefining Geometrical Exactness: Descartes’ Transformation of the Early Modern

⁶⁶ William R. Shea gives a helpful, general statement of the problem using modern notation: “There are given \( n \)
straight lines. From a point \( c \), lines are drawn making given angles with the given lines. If \( n = 3 \), the ratio of the
product of two of the lines from \( c \) to the square of the third is given. If \( n \) is even and greater than two, the ratio of
the product of \( n/2 \) of the lines from \( c \) to the product of the other \( n/2 \) lines is given. If \( n \) is odd and greater than three,
the ratio of the product of \( (n + 1)/2 \) of the lines to the product of the other \( (n-1)/2 \) lines together with a given line is
given. It is required to find the locus of \( c \)”; William R. Shea, *The Magic of Numbers and Motion: The Scientific
In presenting his general approach to solving this problem at the end of Book 1, Descartes focuses on a version in which there are four given lines, but he emphasizes that the solution offered can be applied to more complex versions of the problem with any number of given lines. He lays out the problem as follows:

Let \(AB, AD, EF, GH, \) etc. [see Figure 3], be several lines given in position,\(^{67}\) and let it be required to find a point, such as \(C\), from which having drawn other straight lines on the given ones, such as \(CB, CD, CF,\) and \(CH\), in such a way that the angles \(CBA, CDA, CFE,\) \(CHG, \) etc., are given, and such that that which is produced by the multiplication of one part of these lines is equal to that which is produced by the multiplication of the others, or else that they have some other given proportion, for this does not make the question more difficult. (382)

\[\]

Figure 3

In accordance with the analytic procedure he advocated earlier, Descartes begins approaching the problem by assuming that it has already been solved. Then, “in order to lessen the confusion of all these lines,” he designates one of the given lines and one of the sought lines as the “principal” lines to which all the others will be related (382-83). In the particular example at hand, Descartes chooses the lines \(AB\) and \(BC\) as his principal lines and names them \(x\) and \(y\)

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\(^{67}\) It is important to note, as highlighted by Smith and Latham, that while these lines are given in position they are not given in length; Smith and Latham, trans., *The Geometry*, 26 n. 42.
respectively. He then extends all the given lines so that they intersect these two principal lines at points E, A, G and R, S, T. This produces a series of triangles that can then be used to relate the various lines to the principal lines via algebraic expressions. Descartes does this by first noting that all the angles of triangle ARB are given. Accordingly, the ratio between the lines AB and BR is also given, which Descartes “supposes [pose] as z to b” (383). Given that AB is x, BR is therefore $\frac{bx}{z}$. The line CR is then $y + \frac{bx}{z}$. Having established this, Descartes can now apply the same approach to triangle DRC: because all the angles of triangle DRC are given, the ratio between the sides CR and CD is also given, which is supposed as z to c, “so that CR being $y + \frac{bx}{z}$, CD is $\frac{cy}{z} + \frac{bcx}{z^2}$” (383). Descartes then follows the same general procedure for triangles ESB, FSC, BGT, and TCH, whereby he finds algebraic expressions—made up exclusively of various knowns and the two unknowns $x$ and $y$—for all the lines of the diagram, including the four lines that had to be drawn from point C (namely, CB, CD, CF, and CH).

Next, Descartes notes that this procedure can be expanded to apply to any number of initially given lines and that it will always lead to algebraic expressions that are made up of (at most) three terms with only two unknowns. Moreover, from this fact it is clear that when any of these lines are multiplied together, the product will contain no higher degree of $x$ and $y$ then the number of lines multiplied.

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68 Here we have our first indication that these principal lines are the original version of Cartesian coordinate axes.
69 By supposition the angle ABC is known and the positions of lines AB and AD are given, which together mean that all of the angles of triangle ARB are determined.
70 To show this using modern notation, rewrite the ratio $AB : BR :: z : b$ as the equation $\frac{AB}{BR} = \frac{z}{b}$ and then substitute $x$ for $AB$ and solve for $BR$. 71 In modern notation, the expression will always be of the form $ax \pm by \pm c$. 
Descartes then shows how to find the point C once the algebraic expressions for all the various lines have been found. This is dependant on the stipulation that there is some equality or proportion between the products of two groups of lines drawn from point C. By setting up the equation that captures this relationship and plugging in the algebraic expressions for the individual lines, one generates an equation with only two unknowns, namely, \( x \) and \( y \), such that one can arbitrarily assume one of the two unknown quantities and then use the equation to find the other. Descartes points out that when the problem is posed for five lines or less, \( x \) will not be of a degree higher than two, so that if a known quantity is taken for \( y \), then \( x^2 = \pm ax \pm b^2 \), in which case “one can find the quantity \( x \) with ruler and compass in the manner explained earlier” (386). Once a corresponding pair of \( x \) and \( y \) lengths are known, they can be used to determine the location of point C, and one can then go on to find an infinite number of such \( x \) and \( y \) pairs, whereby they will determine an infinite number of different points that satisfy the same requirements as point C. These points can then be used to describe a curved line, which is the locus that was originally sought.

Descartes goes on to conclude Book 1 by delineating more complex classes of the Pappus problem according to how many lines are initially given, the corresponding degree of the equation, and the geometric means by which the locus can be constructed. For our purposes, however, the most important part of the procedure just outlined is Descartes’s introduction of the

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72 In the particular example under consideration, Descartes has left this stipulation indeterminate, saying simply that there needs to be some such equality or proportion. When he takes this problem up again in Book 2, however, he adds the stipulation that the product of CB and CF is equal to the product of CD and CH (398).

73 This is the case because there will be at most three lines being multiplied together on one side of the equation, but \( x \) is not used in the expression of one of the lines (namely, the first, which is simply \( y \)), so that there will only ever be two \( x \)’s entering into one multiplication.

74 This equation written in modern notation is not far from Descartes’s original. He writes “\( xx = + or - ax + or - bb \)” (although it should be noted that I have here substituted the modern equals sign for the archaic one used by Descartes).
principal lines. These lines are integral to the solution of the problem insofar as their algebraic expression is the basis for the algebraic expression of all the other lines of the problem. In this way, they provide the very frame of reference in relation to which all the other parts of the problem are articulated. It is also important to note that these principal lines are themselves symbolic in that they are instances of Descartes’s new mathematical object (as discussed above in § 3). This can be seen from the fact that both as lines in the diagram (namely, AB and BC) and as algebraic letter signs used to designate those lines (namely, x and y), these principal lines are themselves of an indeterminate quantity that can be given an infinity of different determinate values, which is in fact precisely what has to be done in order to solve the problem. In other words, Descartes’s solution to this problem consists in generating, via the principal lines, an articulation of the general relationship between the various lines of the problem—a relationship that is encoded in both the algebraic expressions and the figural diagram—and then specifying determinate instances that fit that general relationship. Insofar as the principal lines are what make that articulation possible at all, they are central to the very possibility of Descartes’s symbolic mathematical procedure for solving this problem.

This procedure, initially sketched out for solving the Pappus problem in particular, turns out to be the general method for solving problems in the Geometry. We can further flesh out the importance of the principal lines for this procedure by turning to the beginning of Book 2, entitled “On the Nature of Curved Lines,” where Descartes develops his criterion for which curves can be admitted into his mathematical science. Here, too, the principal lines are of central importance.
Descartes begins laying out his criterion of admissibility by criticizing two ancient classifications of curves, that between plane, solid and linear problems, on the one hand, and that between geometrical and mechanical curves, on the other.\textsuperscript{75} The former classification is according to the means required to construct a problem: straight lines and circles for plane problems, conic sections for solid problems, and more complex curves for linear ones. The latter classification is based on the former, grouping the lines used to solve plane and solid problems (i.e., straight lines, circles, and conic sections) as geometrical and those used to solve linear problems (i.e., any lines of greater complexity than the conics) as mechanical.

In contrast to these ancient distinctions, Descartes wants to redraw the boundary between the geometrical and mechanical domains, while also introducing a wider array of classifications within the domain of the geometrical. To do so, he advances a number of arguments against the ancient division between the geometrical and the mechanical, with the goal of showing that this distinction did not correctly delimit which curves were exactly knowable and therefore should be admitted into the mathematical science of geometry. In opposition to this, he lays out his own position as follows:

It is very clear, it seems to me, that if we take Geometry as that which is precise and exact (as one does) and Mechanics as that which is not, and if we consider Geometry as a science which generally teaches how to measure all bodies, one should no more exclude more complex lines than the more simple ones, provided that one can imagine them to be described by a continuous movement or by several movements which are successive and of which the latter are entirely regulated by those which precede it; for, by this means, one can always have an exact knowledge of their measure. (389-90)

We see here that Descartes understands geometry to be concerned with what is “precise and exact” and that any line—regardless of its degree of complexity—meets this criterion if it is

\textsuperscript{75} Molland has shown that Descartes’s presentation of the ancients here is historically inaccurate and seemingly self-serving; see Molland, “Shifting the Foundations,” 35-36.
described by a completely regular or regulated motion, as that guarantees that its measure is exactly knowable. On the other side of the divide are those curves that “truly belong only to Mechanics,” which are “described by two separate motions which have no relation (raport) between them that can be measured exactly” (390). The key issue in the distinction between geometry and mechanics, then, is whether a line can be constructed and known exactly. It is important to note in conjunction with this that this distinction does not create a division between an ideal and material domain, but rather between the exactness versus inexactness of what is bodily, indicating that Descartes understands his geometry to be concerned with what is corporeal.

Descartes goes on to show how encompassing his understanding of the geometrical domain is by giving an example of an instrument that describes an endless series of increasingly complex curves, all of which are completely determined and therefore can be conceived clearly and distinctly. He then further specifies what makes a curved line geometrical:

in order to comprehend together all those [curves] that exist in nature and to distinguish them by order into certain classes (genres), I know of nothing better than to say that all the points of those [curves] that can be called Geometrical, that is to say, those which fall under some precise and exact measure, necessarily have some relation (rapport) to all the points of a straight line, which can be expressed by some single equation for all the points. (392)

Here Descartes reiterates that a geometrical line must have a “precise and exact” measurement, but this time he specifies that this entails that all the points of such a line have a given relation to the points of some other straight line, a relation that is expressible by means of an algebraic

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76 This is what Lachterman calls Descartes’s “principle of kinematic determinateness” (as discussed above in § 2).
77 An example of such a curve mentioned by Descartes is the quadratrix. The construction of this curve depends upon a combination of straight and curvilinear motion, which makes it unknowable to Descartes because he believes that “the proportion that exists between straight and curved lines is not known, and I even believe that it is not able to be known by men, such that one can conclude nothing about this that is exact and assured” (412).
equation. From our consideration of the Pappus problem above, it is clear that this is a reference to the ability to articulate a curved line in relation to some principal lines, which is what allows for it to be algebraically expressible in equation form. A little later, Descartes repeats the claim that all geometrical lines are expressible by algebraic equations, saying that “in whatever way one imagines the description of a curved line, provided that it is among the number of those that I name Geometrical, one will always be able to find an equation to determine all its points” (395). It seems, then, that Descartes has given a second criterion of admissibility, namely, that a line be expressed or determined by an algebraic equation.

These two statements of Descartes’s criterion for the admissibility of curves (that they be generated by a completely regulated motion and that they be algebraically expressible) turn out to be integrally connected in that the complete regulation of the motion generating a curve guarantees that there is an algebraic expression that articulates the relationship between the points of that curve and some straight line (which serves as one of the principal lines). The account of this connection, however, occupies Descartes for the rest of the Geometry, and we cannot consider the arguments for it here. It should be noted, however, that related to this question of the twofold articulation of the criterion of admissibility is the question of which means of construction guarantee the exactness of the lines constructed. Throughout the Geometry, Descartes considers three different means of constructing curved lines—the tracing of lines by means of instruments, the plotting of points according to equations, and the tracing of lines by means of string—and he accepts some form of all three means, while also rejecting
This shows how important it is to Descartes that the geometrical objects of his mathematical science actually be drawn, or at least be capable of being drawn, which serves as an indication that he intends what he is doing in the *Geometry* to be practically instantiated in the physical world.

Most important to the our present concern is that Descartes’s criterion for the admissibility of curves entails that any geometrically constructible line must be described by a completely regulated motion, which entails that it is algebraically expressible, which entails that it can be articulated by means of and in relation to the principal lines. Thus, all the lines that Descartes admits into his geometry can be articulated in relation to the principal lines, and it is in fact only such lines that are admitted. This shows the fundamental importance of the principal lines in that they serve to delimit and articulate the very domain of the geometrical. Moreover,

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For a detailed analysis of these issues, see Bos, *Redefining Geometrical Exactness*, 335-54. According to Bos, Descartes’s arguments relating these various matters, although ultimately inconclusive, “form the deepest and most impressive part of the intellectual effort that produced the *Geometry*” (ibid., 336). He gives a concise presentation of the interconnections between these various parts of Descartes’s mathematical program in the following chart (ibid., 337):

<table>
<thead>
<tr>
<th>Acceptance</th>
<th>Equations</th>
<th>Rejection</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Tracing by motion</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>A. Curves traced by coordinated continuous motions are acceptable in geometry. In particular: if lines or curves are traced continuously one by the other, then the curves traced by their intersections are acceptable in geometry.</td>
<td><strong>B. Curves that are acceptable in geometry according to A.</strong> have algebraic equations; several examples show how the equation can be deduced from the method of tracing.</td>
<td><strong>C. If the separate motions involved in the tracing process are not coordinated, i.e., if they have no measurable relation, then the resulting curve has to be rejected.</strong></td>
</tr>
<tr>
<td><strong>Pointwise construction</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>D. If there is a pointwise construction of the curve for which in principle any of the curve’s points may occur among the constructed ones (generic pointwise construction), then the curve can also be generated by coordinated continuous motion.</td>
<td><strong>E. The algebraic equation implies a generic pointwise construction of the curve (presupposing a complete technique of constructing roots of polynomial equations).</strong></td>
<td><strong>F. If there are points on the curve that cannot occur among those provided by the pointwise construction, then the curve has to be rejected.</strong></td>
</tr>
<tr>
<td><strong>Tracing by procedures involving strings.</strong></td>
<td><strong>H. [Descartes made no remark on the equations of these curves.]</strong></td>
<td><strong>I. If a curve tracing process involves strings that during the process change from straight into curved, then the curve has to be rejected.</strong></td>
</tr>
</tbody>
</table>

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insofar as that domain is occupied only by the symbolic constructions of Cartesian geometry, and that the principal lines themselves are symbolic, it can be concluded that the principal lines serve to define Descartes’s geometrical domain as a symbolic one. This, finally, prepares us to look at the concept of space at work in the Geometry.

§ 6. Space in the Geometry

The word “espace” is used only four times in the Geometry. The first two instances come close together in a passage in which Descartes, citing Propositions 11-13 of Book 1 of Apollonius’s Conics, discusses the characteristic properties of conic sections. In relating his analytic procedure to traditionally-conceived conics, Descartes speaks of “forming a space” (composant un espace) from various parts of the conic (404) and of “subtracting a space” (oster un espace) which has a given relationship to various parts of the conic from a given rectangle (405). In both instances, it is clear that the space to which Descartes refers is the amount or extent of a rectilinear figure defined by two lines and understood in the traditional geometrical manner. He is thus employing the word “space” here to describe the area contained by a figure. Note that in doing so, Descartes is using two lines to articulate and delimit an area contained by those two lines and it is this region that he calls a “space.”

In the third instance of “espace” in the Geometry, Descartes is again referring to an area, although this time it is the area contained by a curve. When listing various properties of curves that can be found after their defining relationships have been expressed via an equation, Descartes claims that this includes “almost everything that can be determined regarding the magnitude of the space which they contain [la grandeur de l’espace quelles comprenent]” (413).
Although he does not expound upon this possibility further in the *Geometry*, it is clear that the conception of space that underlies this claim is of a piece with the first two instances of this word. Of course, the space under a curve is different from the space contained by a rectangle in that it can be open-ended, but the general understanding of space as a region marked off by boundary lines is the same in all three instances. It is also noteworthy here that Descartes speaks not simply of “space” but “the magnitude of space,” which indicates that space is being conceived in such a way that it has a variable property, namely, size or extent.\(^{79}\)

In the fourth and final mention of “*espace*” in the *Geometry*, Descartes invokes a broader conception of space. This comes in the concluding paragraph of Book II, where he says that up to that point he has only spoken about “curved lines which can be described on a flat surface,” but that it is easy to transfer what he has said “to all those [curves] which can be imagined to be formed by the regular movement of points of some body in a space which has three dimensions [*dans un espace qui a trois dimensions*]” (440). Descartes gives only a sketch regarding how to do this, but the heart of the procedure lies in constructing planes at right angles to each other so that all the points of the three-dimensional curve can be related to a line shared by those planes, which ensures that an equation of the curve can still be attained through the use of principal lines. From this it is clear that in extending his mathematical procedure to three-dimensional objects, Descartes also extends the governing role played by his criterion of admissibility, which we have seen to be intimately connected with his use of the principal lines, to the three-dimensional domain.

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\(^{79}\) The idea of space having properties will be important in the following chapters when discussing Descartes’s substantialization of space.
As for the understanding of space expressed in this fourth instance, note first that Descartes is contrasting curves described on a flat surface with those generated by a body moving in a space of three dimensions; this highlights the container-like conception of space here, in that bodies and curves exist within the confines of this space. Note also that Descartes speaks of “a space,” implying that there is a plurality of such spaces, which indicates that space is here conceived as a particular region in which some body or curve exists, but also note that there are no lines that mark off the limits of this space, in that the space is identified by an object within it rather than by any extremities, which indicates that the space itself is unbounded. Finally, note that this space is said to have three dimensions, indicating again that it is conceived of as having a property, although here the property seems to be the capability of containing objects that have length, breadth and depth (as opposed to a flat surface which can only contain an object with length and breadth). 80

The understanding of space invoked in this final instance of the word “espace” is thus the very domain in which some geometrical bodies exist. This is the broadest conception of space expressed in the *Geometry* and it thus seems to govern the other three instances, which retrospectively appear to be more specific or confined versions of this broader conception.

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80 Descartes does not explain how he conceives the three-dimensionality of space in the *Geometry*, and every other instance of the word “dimension” that I have found in this work refers to the degree of an algebraic letter sign or an equation. Given that this meaning of “dimension” can’t be applied to space itself, and that the idea of three-dimensional space as articulated by three coordinate axes is only a later development of Descartes’s mathematics, we are left without a direct indication of how he understands this in the *Geometry*. As mentioned in the previous chapter (Ch. 2, § 7), Descartes defined “dimension” in Rule 14 as “a mode or aspect according to which some subject is considered measurable,” and he there mentioned length, breath, and depth as his first examples (AT X, 447). Elsewhere, he refers to “extension in length, breadth and depth” as the “three dimensions” of matter (Letter to Chanut, 6 June 1647 – AT V 52). While we cannot simply import these understandings of bodily dimension into the *Geometry*, I think it is safe to take Descartes’s understanding of a three-dimensional body to be a body that has length, breadth and depth, from which we must then try to unpack the idea of a three-dimensional space in which such bodies exist.
Common to all four instances, however, is the idea of an area or domain defined by some line segments, although that defining occurs in admittedly different ways. In the broadest conception of space the defining occurs via the principal lines, which are used as the reference in relation to which everything within a space is demarcated, while in the narrower conception of space the defining is achieved by the particular lines that delimit some specific region within that general domain of space.

§ 7. The Symbolic Space of Descartes’s *Geometry*

We are now in a position to examine the symbolic nature of Descartes’s concept of space in the *Geometry*. In doing so, we are confining our discussion to the mathematical domain and thus are only examining his understanding of geometrical space. As we have seen, in its broadest conception this space is the very domain of Descartes’s geometrical objects, which are nothing but exactly measured bodies. Given that this domain is governed by the criterion of admissibility discussed above, the range of objects that exist within this space is restricted to those objects which are generated by completely regulated motions, are algebraically expressible, and can be articulated by means of and in relation to the principal lines.

As already mentioned, this space can first be characterized as symbolic insofar as the mathematical objects that exist within it are themselves symbolic, for in this way Descartes’s geometrical space is a domain made up of nothing but symbolic objects. Given the intimate connection between the principal lines and the criterion of what can be admitted into this space,

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81 Descartes’s understanding of physical space will be examined in the next chapter’s discussion of the *Principles of Philosophy*. 
we can now add that it is the principal lines themselves that serve to delimit the domain of symbolic objects existing within this geometrical space in that they establish the parameters within which and in relation to which such objects have their symbolic meaning.

Beyond this, however, we can now explain how the principal lines serve to symbolically represent this geometrical space itself. As lines given in position but indeterminate in length, the principal lines determine the domain and frame of reference within which all the other geometrical objects of a problem exist. In doing so, the principal lines serve to define a geometrical space not only by giving everything within it its meaning but also by marking it out and visually depicting it on the page. This visual depiction occurs by drawing lines of determinate length that are taken to stand for indeterminate quantities that are capable of taking on an endless range of determinate values.\textsuperscript{82} Insofar as it is the very ability of the principal lines to take on this range of values that defines the domain in which all the other geometrical objects under consideration exist, the figural depiction of that domain by means of those lines serves as a visual articulation of that geometrical space. This amounts to a symbolic representation of that space in that the determinate depiction by means of the principal lines is taken to be the indeterminate range for which that depiction stands, whereby that depiction is taken to be the geometrical space itself.

Moreover, this symbolic representation of space is a further development of the symbolic concept formation that underlies the modern symbolic concept of number. This is the case, first, insofar as it stems from a movement from symbolic mathematical objects to the domain in which those objects exist. Second and more fundamentally, however, the concept of space at work here

\textsuperscript{82} In this way the principal lines are themselves symbolic geometrical objects (as highlighted earlier) in that they are determinate representations of indeterminate quantities in a general relationship.
is a product of the same symbolic abstraction that gives rise to the symbolic concept of number, as can be seen from the following. In depicting space on the page by means of the principal lines, the extended character of geometrical objects is used to represent the geometrical domain of extended objects; in this process, the general character of extendedness is given a figural representation by means of determinate lines, which is then taken to be the very space in which the extended objects of geometry exist. In the language of Rule 14, this involves the coupling and conflating of an indeterminate abstraction existing in the pure intellect—an abstraction which could be called “mere extension,” “extension itself” or “extendedness as such”—with a determinate representation made up of extended line-lengths existing in the imagination and externalized before the eyes.  

It is in this way, then, that the figural depiction of the principal lines on the page symbolically represents the geometrical domain or space that those lines serve to define. Insofar as symbolic abstraction underlies this concept, we can now conclude that the mathematical cognition of Rule 14 is at work in the *Geometry* not only in its symbolic mathematical object but also in the symbolic concept of space that is understood to be the domain of such objects. From this, in turn, we can draw the further conclusion that everything from the *Rules* that it is built into

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83 Klein describes this symbolic concept of space as “the symbolic illustration of the general character of the extendedness” of geometrical objects, which he equates with the modern notion of “Euclidean Space” understood as “the domain of symbolic exhibition by means of line-segments, a domain which is defined by a coordinate system”; Klein, “World of Physics,” 21. It is important to note, however, that Descartes’s principal lines are always a part of the diagram under consideration; they are never posited as a frame of reference separate from the rest of the figure, within which that figure exists. In other words, Descartes never begins with coordinate axes and then places objects into the frame of reference marked out by those axes, as is done with the Cartesian coordinate grid in later developments of Cartesian geometry. While it is easier to see the symbolic representation of space in these separated coordinate axes, the above is meant to show that that symbolic representation is already present in Descartes’s use of the principal lines.
the conceptual structure of the *Geometry*’s mathematical objects is also built into the structure of its concept of space.

With this, we have uncovered the symbolic nature of Descartes’s mathematical concept of space. In the next chapter we will turn to an examination of the physical concept of space contained in Descartes’s *Principles of Philosophy*. By connecting that physical account with the mathematical account discussed in this chapter, we will then be able to show how Descartes brings the mathematical and physical realms together in a way that allows for the application of symbolic mathematics to physical bodies, whereby he creates a conceptual framework for mathematical physics.
CHAPTER 4

The Concept of Space in Descartes’s *Principles of Philosophy*

§ 1. Introduction to the Present Analysis of the *Principles*

The *Principles of Philosophy* is a late but unfinished work in which Descartes presented his philosophy in a systematic fashion in the style of a scholastic textbook.¹ In doing so, he recapitulates much of the material contained in his other works, both published and unpublished, although he reworks and adds to that material throughout. In Part I, Descartes lays out the epistemological and metaphysical principles upon which the rest of his philosophy is based; in the course of this, he covers much of the material that is contained in the *Meditations on First Philosophy*, including radical doubt and the cogito as the starting point for all certain knowledge. In Part II, he discusses the nature of the corporeal world and its motion, as well as the knowledge we can have of them. Part III contains a discussion of the cosmos at large, while Part IV contains a discussion of the earth in particular as well as various terrestrial phenomena, such as magnets and the tides. Throughout Parts II, III, and IV, Descartes discusses a range of topics that are also covered in his other works, including many that are contained in his unpublished

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¹ The *Principles of Philosophy* was published in 1644, containing four of six originally planned parts (as discussed in the following footnote). The original Latin text of the *Principles* is contained in Volume VIII-1 of Adam-Tannery, to which all parenthetical citations in the main text of this chapter refer. Descartes approved a French translation of this work, which appeared in 1647, that contained material not found in the original 1644 edition, most notably a lengthy Preface (this translation is contained in Volume IX-2 of Adam-Tannery). Each Part of the *Principles* is divided up into short Articles, to which I will refer by giving Part and Article number in the following fashion: I.1 (= Part I, Article 1). I have consulted translations of this work from the following editions: *Principles of Philosophy*, trans. Valentine Rodger Miller and Reese P. Miller (Dordrecht: D. Reidel Publishing Company, 1983); *The Philosophical Writings of Descartes*, vol. 1, trans. John Cottingham, Robert Stoothoof and Dugald Murdoch (New York: Cambridge University Press, 1985); *The Philosophical Works of Descartes*, vol. 1, trans. Elizabeth S. Haldane and G. R. T. Ross (New York: Dover Publications, Inc., 1955). For a discussion of the relationship between the *Principles* and the scholastic textbook tradition, see Stephen Gaukroger, *Descartes’ System of Natural Philosophy* (Cambridge: Cambridge University Press, 2002), 32-63.
work *The World*. The projected Part V was to deal with plants and animals, while Part VI was to cover human beings.²

The goal of the present chapter is to uncover the physical understanding of space contained in the *Principles* and to relate it to the mathematical understanding of space uncovered in the previous chapter. The discussion of space is primarily contained in II.1-18 of the *Principles* and our investigation will largely be confined to an analysis of these articles.³ This will begin with a discussion of the nature of body and matter presented in II.1-4 (§ 2), which will then be followed by a discussion of the relationship between matter, substance, extension and quantity in II.5-9 (§ 3). Our textual analysis of the *Principles* will then culminate in a discussion of the physical understanding of space laid out in II.10-18 (§ 4). The main conclusions of this chapter will be then be drawn by relating this understanding of space, first, to the account of abstraction in Rule 14 (§ 5) and then to the mathematical understanding of space contained in the *Geometry* (§ 6). The chapter will then conclude with a discussion of the limited role of space in Descartes’s mathematical physics (§7).

§ 2. The Nature of Body and Matter in *Principles* II.1-4

Part II of the *Principles* is entitled “On The Principles of Material Things” (40). It begins with an article arguing for the existence of material objects. This is necessary because of the

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² Descartes mentions these projected Parts in IV.188, saying “I would not add more to this fourth part of the *Principles of Philosophy*, if, as I previously intended, I were still going to write two more parts, namely, a fifth about living things, or about animals and plants, and a sixth about man” (315). He then goes on to say that he decided not to write these last two parts because he did not want to delay the publication of the first four. Gaukroger has tried to reconstruct the contents of the projected last two parts from various other works of Descartes, including the *Treatise on Man* (which is itself a follow-up to *The World*), *The Description of the Human Body*, and *The Passions of the Soul*; see Gaukroger, *Descartes’ System of Natural Philosophy*, 2-3 & 180-246.

³ There is a similar account of space contained in *The World*, but the discussion of space in that work is much less extensive, for which reason we will only examine the account contained in the *Principles*.
methodological doubt that Descartes employed in Part I of the *Principles*, which rendered the existence of material things doubtful. Here, at the beginning of Part II, he restores the certainty of such existence through an appeal to the previously argued claim that God is not a deceiver. The argument of II.1 begins with the claim that all our sensations come from something different than our minds, as evidenced by the fact that we cannot control them. Within these sensations, however, we clearly and distinctly perceive the existence of some matter that has extension in length, breadth and depth, as well as shape and motion. This perception must therefore be caused by something that has these features and is distinct from both God and our mind, otherwise God would be a deceiver. Thus, Descartes says, “it must be surely concluded that there exists some thing extended in length, breadth and depth, and possessing all those properties that we clearly perceive to pertain to extended things; and it is this extended thing that we call ‘body’ or ‘matter’” (41).

In Article 2, Descartes draws the further conclusion that there is a particular body that is more closely conjoined with our mind than any other body. This is based on the fact that the mind is aware of sensations that cannot come from or belong to the mind by itself, but rather can only belong to it insofar as it is joined to something extended and moveable, namely, the human

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4 In I.4, Descartes began his methodological doubt by saying, “Now, then, since we are concerned only with finding the truth, we will doubt first that any sensible or imaginable things exists” (5).
5 This is argued in I.29 from the fact that God is “supremely truthful and the giver of all light,” from which it is concluded that he neither deceives us nor causes us to err (16). It is then argued in I.30, based on this conclusion, that “the light of nature, or that faculty of knowing given to us from God, can never attain (attingere) any object which is not true, insofar as it is attained by it, that is, insofar as it is perceived clearly and distinctly” (30). From this, the further conclusion is then drawn that “if we notice something clear and distinct in our senses (whether in waking life or in sleep), and we distinguish it from that which is confused and obscure, we will easily recognize what should be taken as true in anything whatsoever” (17). It is precisely this task of identifying what is clear and distinct in our sensory experience of material things that Descartes takes up in Part II of the *Principles*, beginning with the argument for the existence of such material things.
6 In I. 53, Descartes asserted that “extension in length, breadth and depth constitutes the nature of corporeal substance” and that “everything else that can be attributed to body presupposed extension” (25), but he does not give the argument underlying these claims until II.4.
body. In Article 3, however, Descartes argues that sensory perceptions cannot be relied upon to tell us what sensed objects are like in themselves, for they ordinarily only show us what is beneficial and harmful to the mind-body composite. He therefore concludes that we must set aside “the prejudices of the sense” and “use only the intellect by attending carefully to the ideas implanted in it by nature” (42).

He then takes up that task in Article 4, arguing that by doing this “we will perceive that the nature of matter or body considered universally [naturam materiae sive corporis in universum spectati] consists . . . only in that it is a thing extended in length, breadth and depth,” and that all other properties are therefore excluded from the nature of matter (42). This can be concluded, Descartes claims, from the fact that all the qualities besides extension that we sense in corporeal matter can be removed from it while that matter itself remains what it is. To argue for this Descartes’s uses the example of hardness, which according to sensation, he says, is no more than the resistance of a body to the motion of our hands when they encounter each other. If, however, “whenever our hands moved toward some certain area, all the bodies existing there were to recede at the same speed at which our hands approached, we would never feel any hardness,” yet those bodies would still retain the nature of body (42). Descartes concludes from this that the nature of body does not consist in hardness, and then he generalizes to say that “all other qualities of this kind that we sense in corporeal matter can be removed from a body while the body itself remains intact, from which it follows that the nature of body does not depend on any of these qualities” (42). Thus, according to Article 4, the nature of body or matter consists in
extension alone. Here, then, we have the mature Cartesian doctrine that the corporeal world is made up entirely of extended matter.\footnote{This doctrine was anticipated in our earlier discussion of Rule 12 (see Ch. 2, § 5), where it was pointed out that the \textit{Rules} recommended it merely as a useful hypothesis, whereas the actual declaration of the doctrine is found only in Descartes’s later works.}

§ 3. The Relationship Between Matter, Corporeal Substance, Extension, Quantity and Number in \textit{Principles} II.5-9

In Article 5, Descartes names two beliefs that might cause one to doubt that the nature of body consists in extension alone, both of which he goes on to reject in the ensuing articles. The first is the belief that the same body becomes more or less extended in rarefaction and condensation without a corresponding change in its quantity. The second is the belief in a vacuum, or, as Descartes puts it, that “we are not accustomed to say there is a body where we understand there to be nothing other than extension in length, breadth and depth, but rather we say there is only space, and even empty space, which almost everyone persuades themselves is pure nothingness \textit{[purum nihil]}” (43).

Descartes responds to the first belief in Articles 6 and 7 by showing that rarefaction and condensation can be explained without claiming that the amount of a body’s extension changes while the amount of its matter stays the same. According to this explanation, which is laid out in Article 6, rarefaction and condensation occur merely by a change in shape that results from a change in the separation of a body’s parts. Thus, a body can have greater or smaller gaps \textit{(intervallum)} between its parts, which grow or shrink as the body is rarified or condensed, but these gaps are always filled by other bodies which flow into and out of those gaps as they grow.
larger and smaller. Throughout such change, the matter and extension of the rarified or condensed body is simply distributed over a greater or smaller area. There is, however, a change in the overall amount of extension, but the extension of the gaps between the parts of the body is properly attributed to the other bodies that are filling those gaps. In this way, then, the extension of the rarified or condensed body does not itself change, nor is there any change in the amount of its matter; rather, the only change is in the arrangement and distribution of its matter.

In Article 7, Descartes claims that this is the only possible explanation of rarefaction and condensation because the alternative explanation requires a change in quantity and extension without a corresponding change in matter, which is impossible because “any addition of extension or quantity without an addition of substance which has quantity and extension cannot be understood” (44). This last point, however, is only argued in the following articles.

In Article 8, Descartes explains the nature of the distinction between quantity, number, and extended substance. According to him, “quantity does not differ from extended substance in reality [in re], but only on the part of our conception [ex parte nostri conceptus],” and the same is true for the distinction between number and the thing numbered (44). To explicate this, Descartes first establishes that a distinction can be made between quantity and extended substance.

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8 Descartes is here drawing upon his earlier discussion, found in I.60-62, of real distinctions (distinctio realis) versus distinctions of reason (distinctio rationis). A real distinction, Descartes says, “properly exists only between two or more substances, and we perceive these to be really distinct from one another from the sole fact that we can clearly and distinctly understand one without the other” (28). A distinction of reason, on the other hand, “exists between a substance and some attribute of it without which it cannot be understood, or between two such attributes of some single substance”; this kind of distinction is recognized from the inability to form a clear and distinct idea of the substance without the attribute in question or of the one attribute without the other (30). It should be noted, however, that this definition of a distinction of reason does not properly apply to the distinction between quantity and extended substance discussed in II.8 because quantity is not an attribute of that extended substance but rather is convertible with its extension, which Descartes has argued to be its very nature. Nevertheless, there is something like a distinction of reason being articulated here, which is presumably the reason for Descartes’s circumlocutory reference to his earlier discussion of this topic.
substance, as well as between number and the thing numbered. This can be seen from the following line of reasoning. Given a corporeal substance that is ten feet in length, we can consider the whole nature of that substance without attending to the measure of its size because we understand that nature to be the same in any part of the substance as it is in the whole; conversely, we can understand the number ten, as well as the continuous quantity ten feet, without attending to this particular substance because “the concept of the number ten” is the same regardless of its referent, while the continuous quantity ten feet can be understood without this particular substance, although it cannot be understood without some extended substance (44). In this way, then, Descartes establishes that quantity can be considered separately from any particular extended substance and that number can be considered separately from any particular numbered thing, which is what it means for these to differ in our conception. In reality, however, these are not actually separable, according to Descartes, because any change in quantity or extension entails an equal change in the substance itself, and vice versa, from which it is clear that quantity, extension and substance are always conjoined. Thus, to say that quantity and number differ from the quantified or enumerated thing only in our conception but not in reality means that they are only separable in our thoughts or mind, but not in the thing itself.\footnote{This conclusion was anticipated in I.58, where Descartes says “when number is considered not in any created thing but only in abstraction or in general \([tantum in abstracto sive in genere consideratur]\), it is merely a mode of thought \([modus cogitandi]\), and the same is true for everything else that we call ‘universals’” (27). Descartes thus suggests that all generalities, abstractions and universals exist only in our minds. This is reminiscent of the account of abstraction advocated in Rule 14, which recommended that one consider as separate what is not separate, while not denying the connection between the two things being separated. The similarity of these two accounts will be discussed below in § 5.}

In Article 9, Descartes claims that any disagreement with the understanding presented above is purely verbal because there is no other way to perceive the underlying issue. For, when people “distinguish substance from extension or quantity, they either understand nothing by the...
name ‘substance’ or they have only a confused idea of incorporeal substance, which they falsely attribute to corporeal substance, while they relegate the true idea of corporeal substance to extension, which they nevertheless call an accident”; such people therefore “express something in words that is entirely different from what they comprehend in their mind” (45).

Thus, by II.9 of the Principles, it is clear that Descartes thinks there is no real difference between matter, corporeal substance, extension, quantity and number because in reality they are always linked such that they come and go together. While he does grant that some distinctions can be made between these, it is clear that caution must be used in making these distinction in order to avoid confusion and that part of this caution consists in recognizing that any such distinctions exist only in our conception and not in the substances itself because it is only in our thoughts or minds that these can be separated.

§ 4. The Understanding of Space in Principles II.10-18

In II.10-18, Descartes responds to the second reason for doubting that the nature of body lies in extension, namely, the belief in a vacuum. It is here, in his rejection of the possibility of a complete vacuum, that he lays out his understanding of physical space. He begins, in Article 10, by saying that “space or internal place and the corporeal substance contained in it do not differ in reality [in re], but only in the way in which they are accustomed to be conceived by us [in modo quo a nobis concipi solent]; for, in fact, the extension in length, breadth and depth which constitutes the space is exactly the same as that which constitutes the body” (45). The difference

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10 In II.5, Descartes mocked the subtlety of those who “distinguish the substance of a body from its quantity, and it quantity from its extension,” but he did not explain the problem with such a distinction there (42).
in our conception arises, Descartes argues, from thinking of the extension of a body as particular and belonging to that body such that it moves and changes with the body, whereas we attribute “a mere generic unity” (unitatem tantum genericam—45) to the extension of space, such that it does not change when the body that fills it changes but rather remains one and the same regardless of what fills it (provided, Descartes adds, that its size, shape and relative position are maintained). Thus, space does not differ from corporeal substance in reality because both consist only of extension, yet it does differ in conception in that space is the idea of extension considered by itself, separated from any particular body, and granted a generic unity such that it is capable of receiving different bodies while remaining one and the same by itself.

Descartes goes on to explicate this further in Articles 11 and 12, devoting the former to the argument for the identity of space and corporeal substance, while the latter is spent expounding upon their difference in conception. In Article 11, Descartes claims that we can “easily know that it is the same extension that constitutes the nature of body and the nature of space” from the following line of reasoning (46). If we consider the idea of some body, such as a stone, and remove from it everything that is not required by the nature of body, such as hardness, color, and heaviness (all of which a stone could lose without ceasing to be a body), all that will remain after this process of extraction is the idea of something extended in length, breadth and depth. This, however, is all that is contained in the idea of space, regardless of

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11 Descartes argues for the claim that a stone could lose these attributes and still be a body as follows: “we may first exclude hardness because if a stone is melted or reduced to a most fine powder, it will lose that [hardness] but it will not therefore cease to be a body; we may also exclude color because we have often seen stones so transparent that there is no color in them; we may exclude heaviness because although fire is extremely light, it is still thought to be a body” (46).

12 This seems to serve as a further argument that the nature of corporeal matter consists in extension alone.
whether it is taken to be full of bodies or empty (vacuum). Thus, both the nature of body and the nature of space consist in extension alone, which confirms that they do not differ in reality.

In Article 12, Descartes continues with the example from Article 11 to further explicate how space and corporeal substance differ “in the way they are conceived [in modo concepiendi]” (46). When a stone is removed from the space or place in which it exists, we think that its extension has also been removed because it is inseparable from that stone, whereas the extension of the place that the stone occupied remains behind to be occupied by some other body, such as wood, water or air. In this conception, “extension is considered in general [in genere] and thought to be the same” such that it is common to the stone, wood, water, air or whatever other body might occupy the space, even to a vacuum if it is possible for the extended area to remain completely empty (46-47). Descartes again adds, however, that it is necessary that the extended area retain its size, shape and relative position.

Thus, according to II.10-12, the idea of space arises from separating extension from a particular body by considering the extension of that body by itself and in general, and then granting it a generic unity such that it remains one and the same regardless of what occupies it, whereby it becomes an extended area capable of receiving various bodies. Such an idea exists only in our thoughts or mind, however, because in reality extension cannot be separated from some extended matter. There is, therefore, no real difference between space and matter or body because they all consist exclusively of extension.

Descartes goes on in Articles 13-15 to further clarify this understanding of space by delineating and distinguishing various meanings of place. The focus of Article 13 lies in specifying how the relative position of a body is determined. In both Article 10 and Article 12,
when Descartes mentioned that the extension of a space must retain its size, shape and relative position, he specified that a space is determined by its position among external bodies. In Article 13 he explains how that determination occurs. He begins by saying that “the names ‘place’ and ‘space’ do not signify anything different from the body which is said to be in that place, but only designate its size, shape and position among other bodies [situm inter alia corpora]” (47). To determine this position, Descartes says, we must look to other bodies which we consider immobile. Yet, insofar as we can consider different bodies immobile, the same thing can be changing and not changing its place at the same time, as the man on the ship remains in one place in relation to the parts of the ship but changes his place in relation to the shore.

Accordingly, if there are no absolutely immobile points in the universe, which Descartes says will be shown to be probable later, it must be concluded that “nothing has a permanent place, except to the extent that it is determined by our thought [a cogitatione nostra]” (47). For our purposes what is most important about this article is the specification that place and space refer to the size, shape and relative position of a body.

In Article 14, Descartes makes a further distinction within this specification by saying that “the names ‘place’ and ‘space’ differ from each other because place more expressly designates position than size or shape, while on the other hand we attend more to these [i.e., size and shape] when we speak of space” (47-48). As evidence of this, he points out that we frequently say that one thing takes the place of another even if it is not exactly the same in size and shape, but we would deny in that situation that it occupies the same space. Moreover, when some body changes position, we say that it changed place, despite the fact that its size and shape remain the same. Finally, when we say that something is in a place, we only mean that it is in a
certain position among other bodies, but when we add that it fills that place or space then we understand it to be of a determinate size and shape.

Descartes goes on to make a further distinction in Article 15 between internal and external place. He first reiterates that space is always extension in length, breadth and depth, but then points out that when it comes to place “we sometimes consider it as internal to the thing which is in place and sometimes as external to it” (48). The former understanding is simply identical with space, Descartes says, but the later can be understood as “the surface which most closely surrounds what is in the place” (48). He then specifies that the surface here is not a part of either the surrounded or the surrounding body, but simply the boundary or common surface between the two.

For our purposes, what is important about theses distinctions made in II.13-15 is that they are merely verbal or conceptual. All that exists in reality is extended corporeal matter, but the extension of that matter can be separated in thought from the body that possesses it and considered in various ways, whereby it is taken as space or internal or external place.

In Articles 16-18, Descartes uses the account of space developed in the foregoing articles to reject the possibility of a vacuum, understood as a region in which no substance exists. He first points out that if something is extended in length, breadth and depth, it must be a substance because “it is entirely contradictory for nothing to be something extended” (49). Accordingly, wherever there is extension, there must be some extended thing which is the substance to which

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13 In I.11, Descartes said that “it is most evident [nottissimum] by the natural light that there are no affects or qualities of nothing, and therefore wherever we find some affects or qualities, a thing or substance will necessarily be found there to which they belong” (8). This was then reiterated in I.52: “we easily come to know [that a substance exists] from any one of its attributes via the common notion that there are no attributes, properties or qualities of nothing. Thus, from the fact that we perceive some attribute to be present, we conclude that some existing thing or substance is also necessarily present to which that attribute can be ascribed” (25).
that extension belongs. Thus, from the fact that a space that is supposedly a vacuum still contains extension, it can be concluded that there must be some substance in that space that possesses that extension because otherwise there would be extension that belongs to nothing, which is impossible. In this way, then, the impossibility of a vacuum is established.

In Article 17, Descartes explains that “in common usage, we are not accustomed to signify by the word ‘empty’ [vacui] a place or space in which absolutely nothing exists, but rather only a place in which there are none of the things that we think should be there,” such as a water pitcher that is said to empty when it is full of air instead of water (49). In a similar way, we tend to say that a space is empty when it contains nothing sensible, despite the fact that it must be full of some matter. We must therefore attend to the true meaning of the words “empty” and “nothing,” Descartes advises, lest we fall into the mistake of taking a space which is only empty of sensible matter to be entirely empty in the sense of a complete vacuum.

Descartes then claims, in Article 18, that not even God could remove the body filling a vessel and leave a complete vacuum behind. This can be seen, he says, from the following. Although there is not a connection between a vessel and the particular body it contains, there is “a very great and entirely necessary connection between the concave shape of the vessel and the extension taken in general [extensionem in genere sumptum] which must be contained in that cavity” (50). This is the case because a cavity cannot be conceived without the extension contained in it, just as a mountain cannot be conceived without a valley and extension cannot be conceived without some underlying extended substance. Yet, if all body were removed from a vessel without any other body taking its place, all the extension would also be removed from the cavity of the vessel such that it would no longer have a cavity at all; instead, the sides of the
vessel would be in contact with each other since there would be nothing between them, not even distance “because all distance is a mode of extension and therefore cannot exist without an extended substance” (50).

Let us unpack this striking argument further. Given a vessel with a cavity, there is an extended space between its walls. The vessel only serves to delimit or define that space, however, because the extension of that space does not belong to the vessel itself but rather to the body filling the vessel that exists in that space. In order to consider the extension between the walls of the vessel as a space by itself (as opposed to considering it as the extension of the body filling the vessel), the extension must be taken in general and granted a generic unity, whereby it is separated from the particular body to which it belongs and then considered by itself in isolation from any particular body, whereby it becomes capable of receiving other bodies while remaining the same regardless of what body occupies it. In this way, the extension between the walls of the vessel is taken to be the space of its cavity, which is common to any body that fills the vessel while also being indifferent to which bodies in particular are filling it. This space cannot be empty, however, because then there would be no body between the walls of the vessel, which would mean that there is no extension between the walls of the vessel because there would be no matter or substance that is extended. In such a situation, the walls of the vessel would simply touch. It can be concluded, therefore, that a complete vacuum is absolutely impossible and, accordingly, that the corporeal world must rather be a complete plenum of extended matter.\textsuperscript{14}

\textsuperscript{14} Descartes’s plenum raises the question of how individual parts of matter can be distinguished within the continuum of extended matter that makes up the whole, which he addresses in II.23 as follows: “All the properties which we clearly perceive in [matter] can be reduced to this one fact, that it is divisible and its parts are therefore
This conclusion provides us with an opportunity to recapitulate the physical understanding of space laid out in the *Principles*. All that exists in the corporeal world is extended matter, and space does not differ in reality from that matter because the nature of both consist in extension alone. There is, however, a certain way of conceiving extension (laid out in the previous paragraph) that gives rise to an understanding of space as an extended region capable of being filled by various bodies while remaining one and the same itself. This understanding arises from separating the extension of a body filling a space from that body and then considering that extension by itself, despite the fact that it cannot actually exist separately from that body. In doing so, the mind treats extension in such a way that it creates an idea of space that differs in conception from the corporeal matter that fills a space, despite the fact that in reality there is no such difference.

§ 5. The Connection Between the Understanding of Space in the *Principles* and the Account of Abstraction in Rule 14

Descartes’s account of space in the *Principles* connects with the mathematical cognition of Rule 14 in that the account of the conceptual difference that distinguishes space from body maps onto the account of abstraction that makes up the first part of the mathematical cognition of Rule 14. As we saw in § 7 of Ch. 2, Descartes advocated a particular understanding of abstraction in Rule 14, which served as the starting point for the account of mathematical cognition contained in that Rule. According to that understanding, abstraction occurs when the movable, and hence that it is capable of all those dispositions [affectio] that we perceive can follow from the motion of its parts”; thus, he concludes, “all the variation of matter, or all the diversity of its forms, depends on motion” (52).
pure intellect separates a general, abstract entity from a subject that it is necessarily contained in, which the intellect then considers by itself as if it existed separately although it in fact does not. Such an entity must be general because, as an object of the pure intellect, it is indeterminate in the sense that it lacks the determinacy that comes from being a particular.

Despite the fact that the account of the cognitive faculties of the Rules is not present in the Principles, the account of abstraction from Rule 14 can still be applied to the understanding of space laid out in II.1-18. For, according to that understanding, space does not differ from body or matter in reality but only in conception. In explaining what this means, Descartes is clear that this concept of space is a product of the mind’s separating an aspect of a substance (namely, extension) and considering it by itself, as if it were separate although it is in fact not. He even mentions that in doing so, extension is taken in general. It is therefore fair to say that the concept of space in the Principles is, in the terms of Rule 14, an abstraction existing only in the mind or intellect.\textsuperscript{15}

While this is not to say that the account of mathematical cognition of Rule 14 is present in the Principles as a whole, it does show that there is room for the employment of that cognition within its understanding of space and the material world. For, insofar as space is an abstraction of extension from corporeal matter, the first half of the mathematical cognition of Rule 14 is present in the Principles, setting up the possibility of the second half, namely, the assigning of a determinate representation to the indeterminate content of the intellect, resulting in the generation of a symbol. In this way, then, the possibility of a symbolic representation of the physical understanding of space contained in II.1-18 is left open in the Principles, which paves

\textsuperscript{15} This description is not completely foreign to the Principles, as evidenced by the discussion of I.56-62 (much of which has been outlined in footnotes above).
the way for a connection between the understanding of space in this work and the understanding of space contained in the *Geometry*.

§ 6. The Connection Between Space in the *Principles* and Space in the *Geometry*

As we have now seen, according to the *Principles*, physical space is the domain of corporeal objects, understood as an extended area in which such objects exist. This understanding of space is only an abstraction in that all that exists in reality is extended bodily matter, from which space is only conceptual (but not really) distinct. As argued in the previous section, this maps on to the first half of the mathematical cognition of Rule 14, thereby preparing the way for the second half of that cognition. For, insofar as the physical understanding of space is achieved by abstracting the extension from extended bodies, the symbolic mathematical understanding of space can then be imposed upon any physical space by using some parts of the extension within that physical space as the determinate representation of that indeterminate abstraction, whereby the symbolic domain of Descartes’s geometry is constructed or instantiated in the physical world.

In other words, insofar as extension is all that exists in the corporeal world, and extension is both that from which the abstractions of mathematics are derived and that which is used to symbolically depict those abstractions, the physical world is capable of being treated as the domain of symbolic mathematics insofar as any physical extension can be made into a symbolic geometrical object simply by designating that extension as the representation of a mathematical abstraction. While this can be done individually with any particular bit of extension, it can also
be done for a space as a whole by designating some lines of extension within a physical space as the principal lines of Descartes’s geometry, whereby that physical space is made into a symbolic geometrical one. In this way, then, symbolic mathematics can be applied to the physical world by transforming a physical space, understood as the domain or container of corporeal objects, into a geometrical space, understood as the domain or container of symbolic mathematical objects, which, in turn, results in a transformation by the mind of the corporeal objects themselves into symbolic mathematical objects in that the physical extension of those objects is thereby used as the determinate representation of an indeterminate abstraction.

Let us return to the example of the vessel from II.18 to spell out more fully how a physical space can be taken as a mathematical space by constructing symbolic objects within it. First, to arrive at a physical understanding of the space contained by a vessel, the particular extension of the area delimited by the cavity of the vessel, which is actually the extension of some particular body filling that cavity, must be considered by itself, separate from any particular body, and granted a generic unity so that it remains the same regardless of what happens to be filling it. This understanding of space is an abstraction in that it depends upon the consideration of the extension of an extended body as if it were separate when it is in fact not. This general abstract concept can then be given a determinate representation whereby a symbolic mathematical object is constructed. In order to do this for the physical space of the vessel as a whole, we can assign two lines of extension—say, a line on the base of the cavity and another on one of its sides—as principal lines that serve to articulate a symbolic geometrical space. In this way, the physical space of the vessel becomes a symbolic mathematical space in that the extension contained within that space can be treated in an exactly knowable and determinate
fashion. Thus, just as the physical space was an extended area capable of receiving various bodies, once it becomes a symbolic mathematical space it is a mathematical domain in which symbolic mathematical objects exist. At that point, we are free to treat the extended corporeal matter within that space as the extended mathematical objects of Descartes’s geometry. The mathematization of physical space therefore allows physical objects to be treated mathematically.

Space thus serves as a bridge between the mathematical and the physical in that the same space can be taken as the domain of both; understood physically, space is a domain or area in which corporeal objects exist, while understood mathematically, it is a domain of symbolic mathematical objects. These two understandings of space are inherently connected via their relationship to extension: a mathematical space contains extension that must be the extension of some extended matter, while physical space is made up of extension that is always susceptible to being treated mathematically in that it can have the symbolic construction of mathematics imposed upon it. In this way, then, the concept of space provides a connection between the mathematical and physical domains in a way that makes the two convertible.

Descartes himself asserts the identity of the physical and mathematical domains in II.64 of the *Principles*, which is the final article of Part II. He says:

> I fully acknowledge that I know of no other matter in corporeal things than that which is divisible, shapeable and moveable in every way, which the geometers call quantity and take as the object of their demonstrations; and [I acknowledge] that I consider absolutely nothing in that [matter] except those divisions, shapes and motions, and that I accept nothing as true about these which is not deduced so evidently from those common notions whose truth we cannot doubt that it must be taken as a mathematical demonstration. And because all natural phenomena can be explained in this way, as will appear in what follows, I think that no other principles of physics are to be admitted or even desired. (78-79)
It is therefore explicitly stated in Descartes’s own account that the matter of geometry and the corporeal world are the same, that everything that can be known about this matter is deduced through mathematical demonstration, that all natural phenomena can be explained through such a mathematical approach, and, finally, that no principles are needed in physics other than those of geometry. Thus, we have here a clear claim by Descartes that all knowledge of the corporeal world is attained through mathematical and geometrical means. Moreover, it should be clear from the foregoing analysis of Descartes’s concept of space that that concept is central to the possibility of Descartes’s making such a claim for a mathematical approach to physics.

§ 7. The Limited Role of Space in Descartes’s Mathematical Physics

We have now completed our goal of uncovering the nature and structure of Descartes’s concept of space. As was just argued, this concept spans and unites the mathematical and physical domains, whereby it allows for an application of modern symbolic mathematics to the physical world. By creating such a possibility, Descartes’s concept of space provides a conceptual framework for modern symbolic mathematical physics. Yet, it is important to note that, although Descartes does make forays into mathematical physics, in doing so he does not employ his concept of space to justify or allow for the application of mathematics to physics. In fact, Descartes never employs the mathematics of his Geometry in any physical context; instead, whenever he brings mathematics to bear on a discussion of the physical world, it is traditional (i.e., non-symbolic) mathematics that he uses. Thus, despite the fact that Descartes contributes

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16 Descartes puts this much more succinctly elsewhere, when he says “my entire physics is nothing other than geometry” (Letter to Mersenne, 27 July 1638 – AT II 268). As we will see in the next section, however, Descartes had only limited success in actually producing a mathematically-informed physics.
to the invention of modern symbolic mathematics, which he expressly understands as a tool for the solving of all problems (including physical ones), he himself never managed to achieve a real synthesis between modern mathematics and physics. As a result, if we wanted to consider an example of the application of symbolic mathematics to the physical world, we would have to turn to an examination of later thinkers.

Nevertheless, the foregoing analysis is intended to have shown that although Descartes himself did not produce a symbolic mathematical physics, he still made an important contribution to the possibility of combining modern mathematics and physics insofar as his concept of space provides a framework within which the physical extension of the world can be treated as a symbolic mathematical object. In explicating this aspect of Descartes’s thought, we have therefore been aimed at contributing to the understanding of the conceptual grounding of mathematical physics that Descartes’s concept of space provides, rather than to the understanding of any mathematical-physical work that has been done upon those grounds. Yet our analysis of the former is intended to aid in the investigation of the latter, and so some appraisal of Descartes’s mathematical-physical works is still called for. It should be kept in mind, however, that it would only be in following up on our analysis by tracing other thinkers’ later developments of the lines of inquiry initiated by Descartes that we might see a full-blown symbolic mathematical physics as made possible by Descartes’s concept of space.

The natural place for us to begin a discussion of Descartes’s mathematical-physical works is in the rest of the *Principles*, as that will serve as a continuation and conclusion of the analysis of the present chapter. Moreover, although the *Principles* does not contain the mathematization of physical space that its account makes possible, its purely physical
understanding of space does significantly shape the account of natural phenomena contained in that work. The following discussion therefore serves as a starting point for investigating the way Descartes’s concept of space can shape a physical inquiry.\footnote{It should be noted that it is generally recognized that there is no developed mathematical physics in the *Principles*. As Alexandre Koyré says: “Le fait est connu. La physique de Descartes, telle que nous la présentent les *Principes*, ne contient plus de lois mathématiquement exprimables. Elle est, en fait, aussi peu mathématique que celle d’Aristote”; Alexandre Koyré, *Études Galiléennes* (Paris: Hermann, 1966), 128.}

After laying out his account of space in II.1-18, Descartes goes on to draw a number of conclusions about his plenum view of the world, such as the fact that it cannot contain indivisible atoms of matter (II.20), that the extension of this world is unlimited (II.21), that the extended matter that makes up this world is the same everywhere (II.22), and that it is only the motion of various parts of the plenum that differentiates those parts and gives them their properties (II.23). In II.24, he then makes it clear that by motion he means only locomotion, which he defines in II.25 as “the transfer [*translationem*] of one part of matter, or of one body, from the vicinity of those bodies which immediately touch it and are considered as at rest, into the vicinity of other bodies” (53). This definition has important consequences both for the understanding of motion itself, such as the fact that all motion is relative (as discussed in II.29-32), and for the kind of motion that can exist in Descartes’s plenum world, such as the fact that all motion entails a complete circle or ring of moving bodies in which the displacement of the first initiates a change that results in the last taking the place of the first (as discussed in II.33).

Throughout these articles, Descartes’s conception of the physical world as a space filled with nothing but extended matter determines not only the nature of that world itself but also the kind of causality that is at work in that world. In doing so, this conception also determines the nature of the explanations of physical phenomena in that world insofar as all those phenomena...
must be caused exclusively by the motion of extended matter. Descartes’s conception of physical space thus determines the very nature of the causal accounts that are open to him, and in this way it shapes his attempts at explaining the physical world in the rest of the *Principles*.

The influence of Descartes’s physical understanding of space continues to be evident in the account of the causes of motion laid out in the remainder of Part II. In II.36, Descartes says that there are two causes of motion, “the universal and primary one, which is the general cause of all the motions in the world, and then the particular ones, by which individual parts of matter acquire motions they did not previously have” (61). The former cause is God, who created the material world with a certain quantity of matter in its parts, which quantity He has maintained ever since. From this original cause of motion, Descartes then deduces three particular causes of motion, which he refers to as the “rules or laws of nature” (62). The first, discussed in II.37, is that “each thing, insofar as it is simple and undivided, always remains in the same state by itself, and never changes except by external causes,” from which it follows that what is at rest stays at rest and what is in motion stays in motion until some other body disturbs it (62). The second law, discussed in II.39, is that a body in motion continues its motion in a straight line so long as it is left to its own devices. The third law, discussed in II.40, governs “all the particular causes of the changes which occur in bodies” and states that “when a moving body meets another, if it has less power [*vim*] to continue in a straight line than the other has to resist it, then it is deflected toward other parts, retaining its quantity of motion and losing only the direction of its motion; if, however, it has more power, then it moves the other body with itself and loses as much of its motion as it gives to that other” (65). Together, these three laws allow Descartes to attempt to determine the rules that govern the behavior of bodies in collision; for, given the
conservation of the quantity of motion guaranteed by the first law, all that is required to determine the motion of colliding bodies is to calculate how much power those bodies have to cause or resist motion, which is found simply by multiplying their size and speed.

Descartes goes on to lay out seven rules of motion in II.46-52. The first stipulates that if two bodies of equal size are moving at equal speed in exactly opposite directions, their collision will result in the bodies rebounding in the opposite direction without having lost any of their speed. The second says that if two bodies of unequal size collide with equal speed, both bodies will retain that speed but the smaller body will rebound in the opposite direction while the larger body continues on in its original direction. The third says that if two bodies of equal size collide with unequal speeds, the body with greater speed will cause the other body to rebound and in doing so transfer one half of its excess speed to the other body, such that both will move off in the same direction with the same speed. The fourth states that a moving body that collides with a larger body at rest will rebound in the opposite direction, having transferred none of its speed to the larger body, which will simply be unaffected by the collision. The fifth says that if a moving body collides with a smaller body that is at rest, the moving body will transfer as much of its motion to the smaller body as is required to leave the two bodies traveling in the same direction with the same speed. The sixth says that if a moving body collides with a body of equal size that is at rest, the moving body will rebound, having transferred one quarter of its speed to the body at rest, which will begin to move in the opposite direction. Finally, the seventh covers three situations in which the bodies are moving in the same direction but with unequal speeds, such that they will still collide when the faster body overtakes the slower: if the slower body is also the larger body, but the smaller body’s speed exceeds the larger body’s speed by an amount
greater than the difference in their size, then the smaller quicker body will transfer as much of its speed as is required to cause them to travel with the same speed in the same direction; if the smaller body’s speed does not exceed the larger body’s speed by an amount greater than the difference in their size, then the smaller quicker body will rebound in the opposite direction while maintaining its speed, and the larger slower body will proceed unaffected; if the difference in size between the bodies is exactly equal to the difference in their speeds, then the faster body will impart some of its motion to the slower body and rebound with the rest.

These rules are notoriously problematic and in some cases completely contrary to experience (such as the fourth rule’s claim that a smaller body can never set a larger body in motion). Nevertheless, they clearly stem from the conception of the physical world laid out in the beginning of Part II. Here too, then, we see that the shape of Descartes’s physics in the *Principles* is determined by his conception of the physical world as a space filled with nothing but extended matter. It should also be noted that the calculations of quantities of motion entailed by these rules is about as mathematical as Descartes’s physics gets in the *Principles*. Insofar as these rules (and their difficulties) stem from Descartes’s peculiar understanding of physical space, we thus have here an indication of the limitations on the mathematical physics of the *Principles* that result from the fact that Descartes’s does not fully mathematize his physical space in that work.

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18 The accounts of physical phenomena found throughout Parts III and IV are almost exclusively descriptive and qualitative, rather than quantitative, and when they are quantitative they are no more mathematically sophisticated than the rules of motion. As Gaukroger points out, “there is no mathematical demonstration of any kind in the *Principia*”; Gaukroger, *Descartes’ System of Natural Philosophy*, 146. For a fuller discussion of the lack of mathematics in the *Principles*, see Daniel Garber, “A different Descartes: Descartes and the programme for a mathematical physics in his correspondence,” in *Descartes’ Natural Philosophy*, ed. Stephen Gaukroger, John Schuster and John Sutton (New York: Routledge, 2000), 113-14.
Moreover, a study of those works in which Descartes does depend heavily upon mathematics in his account of physical phenomena (such as *The Optics* and *The Meteorology*) shows that even when he does employ mathematics to explain physical phenomena, he still does not employ the symbolic mathematics of his *Geometry*. In other words, such a study shows that Descartes’s mathematical physics is never symbolic mathematical physics. From this fact, it can be concluded that Descartes never brought about the complete synthesis of the domain of symbolic mathematics and the domain of the physical world that his concept of space makes possible (as laid out above). For, as we have seen, the complete mathematization of physical space depends upon a combination of the mathematical understanding of space found in the *Geometry* and the physical understanding of space found in the *Principles*—a combination which Descartes himself never managed to realize. Thus, Descartes’s concept of space is never actually used to justify the application of symbolic mathematics to the physical world within his own works. For this reason, the foregoing analysis of Descartes’s concept of space can only safely be taken to speak to the possibility of a conceptual grounding or framework for mathematical physics, rather than any actualization of that possibility. Whether such a possibility is in fact ever realized can only be assessed by tracing the relevant lines of development that have their roots in the thought of Descartes beyond his work into that of his inheritors.
CHAPTER 5

Conclusion

§ 1. Descartes’s Concept of Space

Having completed our analyses of the *Rules*, the *Geometry* and the *Principles* in the previous three chapters, we are now in a position to highlight our major conclusions regarding Descartes’s concept of space. First, as was discussed at the end of Chapter 3 (§ 7), insofar as this concept is mathematical, it is a product of the same symbolic abstraction that underlies Descartes’s symbolic mathematical objects. This means that this concept is predicated upon the mathematical cognition of Rule 14, which is, in turn, predicated upon the particular mind-world relationship laid out in Rule 12. From this, the important conclusion can be drawn that Descartes’s symbolic-mathematical concept of space is dependent upon the operational or methodological dualism of the *Rules*.

Furthermore, as was laid out in the previous chapter (§ 6), Descartes’s concept of space spans and unites the mathematical and physical domains, thereby allowing the symbolic character of Descartes’s mathematics to make its way into the conception of the physical world. As a result of this, the dualism that underlies Descartes’s mathematics can also make its way into the conception of the physical world, at least insofar as it is treated mathematically. Yet, as we also saw, Descartes himself claims that the mathematical and physical domains are entirely the same, from which it can be concluded that this dualism underlies his conception of the physical world as a whole. This dualism is therefore at play in any attempt at gaining knowledge of the corporeal world that depends upon Descartes’s combination of mathematics and physics.
Moreover, we have also seen that Descartes’s concept of space is central to his attempt at combining mathematics and physics in pursuit of knowledge of the corporeal world. This is the case for both the mathematical and the physical sides of this endeavor. Space is foundational to the mathematical side of this endeavor in that it confines the range of the mathematical domain to those objects which can be known in an exact and determinate manner. It is foundational to the physical side of this endeavor in that it allows the entire domain of corporeal objects to be taken as susceptible to mathematical knowledge; for it is only by moving to the general domain of corporeal objects that the corporeal domain as a whole can be understood to be completely convertible with the mathematical domain. Thus, it is the abstractive generalization that underlies Descartes’s physical understanding of space—an abstraction which moves from the extension of particular bodies to an extended area filled with nothing but extended bodies—that allows all corporeal bodies to become objects of mathematical knowledge.

Here we have confirmation that the account of the mind and its operations found in the *Rules* is central to both Descartes’s concept of space and to that concept’s role in uniting the mathematical and physical domains. For, as our analysis of the *Geometry* has shown, the mathematical domain is itself dependent upon the mathematical cognition of Rule 14, while, as our analysis of the *Principles* has shown, the corporeal domain is determined by an abstraction that constitutes the first half of that cognition while leaving open the possibility of the second half. The combination of the mathematical and physical domains effected by Descartes’s concept of space thus inextricably depends upon and involves this particular mathematical cognition as well as its underlying epistemological basis. Any attempt at developing a
mathematical physics that depends upon that concept is therefore also predicated upon those foundations.

With this last point, we have reached our major conclusion regarding Descartes’s concept of space. It is a central concept that connects a number of different aspects of Descartes’s thought in that it is founded upon a certain conception of the mind and its operations, while being, in turn, the foundation upon which mathematics and physics can be combined into a mathematical-physical science. Given that the inauguration of such a science is one of the main goals of Descartes’s writings, it can therefore be concluded that his concept of space, along with all its conceptual presuppositions, holds a pivotal place within Descartes’s philosophy as a whole. The major contribution of the foregoing analysis to the understanding of this concept has been to clarify those conceptual presuppositions in such a way that reveals the symbolic nature of this concept of space.

§ 2. Klein on Descartes and His Concept of Space

We now return to the larger framework within which our investigation has taken place by revisiting Klein’s discussion of Descartes. According to Klein, what “gives [Descartes] his tremendous role in the history of the origin of modern science” is that “he was the first to assign to ‘algebra’ . . . a \textit{fundamental place in the system of knowledge in general}”; in doing so, he created a “\textit{symbolic discipline}”—a “science which aims from the first at a comprehension of the totality of the world”—that “slowly broadens into the system of modern mathematical physics.”\footnote{Klein, \textit{GMTOA}, 184.} As mentioned in our discussion in Chapter 1 (§ 8) of the central place Descartes holds in Klein’s
analysis, Descartes contributes to the development of modern science in three distinct areas: mathematics, philosophy, and natural science itself. As a mathematician, he contributes to the development of modern mathematics by extending the newly invented symbolic algebra to the realm of geometry; as a philosopher, he develops the epistemological underpinnings for the new symbolic mathematics, while also providing a conceptual framework within which that mathematics can be applied to the physical world; as a scientist, he made important contributions to the mathematical study of nature, such as his analysis of the rainbow and his investigations into the nature and constructions of lenses.  

While our investigation has been confined to an analysis of the first two of these three areas of contribution, our findings nevertheless serve as a propaedeutic to an analysis of the third. For it is only by understanding how Descartes’s mathematical and philosophical contributions work together that one can accurately assess his contributions to the development of natural science. In Klein’s words:

Descartes’s great idea . . . consists of identifying, by means of “methodological” considerations, the “general” object of [algebra]—which can be represented and conceived only symbolically—with the “substance” of the world, with corporeality as “extensio.” Only by virtue of this identification did symbolic mathematics gain that fundamental position in the system of knowledge which it has never since lost, even though Descartes may have failed subsequently in working out completely his original conception.”

On Klein’s account, then, it is precisely the identification of the object of symbolic mathematics with the object of physics that allows for the development of mathematical science in its modern form. The qualification that Descartes “may have failed . . . in working out completely his

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2 For Descartes’s discussion of the rainbow, see the Eighth Discourse of the Meteorology. For his discussion of lenses, see the Seventh through Tenth Discourses of the Optics, and the latter part of the Second Book of the Geometry.

3 Klein, GMTOA, 197-98.
original conception” serves as an important reminder that we are here engaged in the task of uncovering conceptual presuppositions, which need not have been actively—or even consciously—developed in order to have taken hold and shaped later developments. Of course, the question of whether such conceptual presuppositions do indeed make their way into such later developments can only be addressed by a detailed study of those later developments, but it is first necessary to uncover what those conceptual presuppositions might be, which is exactly what the present study has aimed to do by highlighting that Descartes’s concept of space plays a central role in connecting the mathematical and physical domains.

In fact, it is a major conclusion of this study that Descartes’s concept of space plays a mediating role between the mathematical and philosophical developments that underlie modern mathematical physics and, therewith, the actual scientific developments made within the modern mathematical sciences. Thus, by uncovering the conceptual presuppositions built into Descartes’s concept of space, we have opened the possibility of evaluating whether and in what way those presuppositions make their way into such later developments. While we are not in a position to take up that task here, we can point ahead to what that task might look like by returning to Klein’s claims about Descartes’s concept of space (discussed earlier in Ch. 1, § 9), which we are now in a position evaluate more fully.

In *Greek Mathematical Thought and the Origin of Algebra*, Klein says the following:

Extension has . . . a twofold character for Descartes: It is “symbolic”—as the object of “general algebra,” and it is “real”—as the “substance” of the corporeal world. More exactly, in Descartes’ thinking, the dignity of representing the substantial “being” of the corporeal world accrues to extension precisely by reason of its symbolic objectivity within the framework of the *mathesis universalis*. Only at this point has the conceptual basis of “classical” physics, which has since been called “Euclidean space,” been created.
This is the foundation on which Newton will raise the structure of his mathematical science of nature.⁴

Here Klein highlights the role that extension plays in Descartes’s uniting of mathematics and physics. On the one hand, extension has a symbolic character in Descartes’s mathematical thinking in that it serves as the object of algebraic geometry (as our analyses of the Rules and the Geometry have confirmed); on the other hand, extension is also taken to be the very substance of the corporeal world (as was confirmed by our discussion of the Principles). Due to this “twofold character,” extension can be taken as the object of both mathematical and physical knowledge. Based on this fact, Klein makes the significant claim that this is the basis of classical (Newtonian) physics, which goes by the name of “Euclidean space.”⁵ While we cannot here evaluate such claims regarding the basis of the later mathematical sciences, our analysis has shown that it is by moving from the mathematical and physical understandings of extension to the general domains of mathematical and physical objects, understood as extended areas in which such objects exist, that the mathematical and physical domains can be taken as entirely convertible. Moreover, it has also been shown that this movement is accomplished by the mathematical cognition that gives rise to the concept of space, and that it is the very equating of the mathematical and physical domains which is made possible by this movement that allows for an entirely symbolic mathematical physics. We can therefore cash out Klein’s claim that the creation of “Euclidean space” serves as the basis for classical physics by understanding this to mean that Descartes’s invention of a new symbolic concept of space, which spans and unites the

⁴ Ibid., 210-11. As mentioned in Footnote 101 of Chapter 1, Klein gives no evidence for his claims regarding Newtonian science.
⁵ Regarding Klein’s equating the modern, symbolic concept of space with the modern understanding of “Euclidean space,” see Footnote 100 of Chapter 1, as well as the discussion of the following paragraph.
mathematical and physical domains, provides a conceptual framework that serves as a
foundation upon which a symbolic mathematical physics can be built.

Klein sheds further light on this topic in a slightly more expansive passage from his
lecture “The World of Physics and the ‘Natural’ World.” There, he says:

Descartes’ concept of *extensio* identifies the extendedness of extension with extension
itself. Our present-day concept of space can be traced directly back to this. Present-day
Mathematics and Physics designate as “Euclidean Space” the domain of symbolic
exhibition by means of line-segments, a domain which is defined by a coordinate system,
a relational system, as we say nowadays. “Euclidean Space” is by no means the domain
of the figures and structures studied by Euclid and the rest of Greek mathematics. It is
rather only the symbolic illustration of the *general character of the extendedness* of those
structures. Once this symbolic domain is identified with corporeal extension itself, it
enters into Newtonian physics as “absolute space.”

Note that here, too, Klein equates the modern symbolic concept of space with the modern
understanding of “Euclidean space,” although in this passage he explicitly adds the important
qualification that this is not the domain of Euclidean geometry. While we cannot evaluate the
Euclidean portion of this claim, our analysis has confirmed that insofar as the modern concept of
space is symbolic, it depends upon the entire conceptual structure of modern symbolic
mathematics. Klein himself seems to indicate as much in this passage by pointing to symbolic
abstraction as an important link between Descartes’s concept of extension and the modern
concept of space. For, in saying that this concept of extension conflates “the extendedness of
extension with extension itself,” Klein highlights that Descartes’s concept of extension depends
upon substituting what has traditionally been understood as a property of substances for such
substances themselves, whereby the substance’s “being extended” or its “extendedness” is taken
as the whole of its being. This concept of extension is therefore halfway to becoming an

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abstraction (in the sense laid out in Rule 14) in that it is concerned with the “the extendedness of extension,” which when considered by itself becomes the abstraction extendedness-as-such. This abstraction can then be taken as the very domain of extension and given a symbolic representation by means of line segments, whereby it is transformed into the modern symbolic concept of space in that it becomes a “symbolic illustration” of the general character of being extended—an illustration that depends upon the conflation of a first-intentional depiction (namely, the line segments or coordinate system) and the second-intentional concept for which it stands (namely, extendedness-as-such). Finally, as Klein indicates with his reference to the role of absolute space in Newtonian physics, insofar as this symbolic domain is understood to encompass the corporeal domain as well, it provides a framework within which mathematical physics can exist.

Thus, in both of these passages in which Klein discusses space, his comments serve as an indication of the importance of Descartes’s symbolic concept of space for the invention of modern mathematical science. It has been one of the main goals of the present study to spell out and defend the parts of these comments that pertain to Descartes. Having done so, the door is now open to follow up on Klein’s broader claims by investigating whether or not the conceptual presuppositions of Descartes’s mathematical-physical thinking make their way into the later developments of modern science. Klein has suggested one way to begin such an investigation, namely, by examining Newton’s concept of absolute space and its foundational role in his mechanics. As a second possibility, I would also suggest looking at the correction of

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7 For Newton’s discussion of absolute space, see the Scholium to the Definitions in his Principia.
Descartes’s “rules of motion” found in Christiaan Huygens’s *On the Motion of Colliding Bodies.*

§ 3. Contributions of this Dissertation

Let us now close with a reflective survey of the contributions made by the preceding analyses. These contributions fall into two categories: contributions to the study and understanding of Descartes, and those to the study and understanding of Klein. Regarding the former, the main contribution has been to the understanding of Descartes’s concept of space and its presuppositions. To uncover the nature of this concept, we first had to clarify the understanding of the mind-world relationship that underlies Descartes’s mathematical thinking, before then spelling out how that thinking is at work in Descartes’s mathematics as well as his mathematical concept of space. Against this backdrop, we then uncovered Descartes’s physical understanding of space and showed how it is related to his mathematical understanding in such a way that allows for a combination of the mathematical and physical domains.

As for the contributions made to the understanding of Klein, the foregoing has served as a model example of the kind of investigation for which Husserl called and Klein began, namely, a historical investigation of the roots of modern mathematical science in its symbolic form. As discussed in Chapter 1, Husserl provided the philosophical justification for the project of desedimentation by articulating a necessary connection between historical and epistemological investigations, while Klein initiated that project by desedimenting the conceptual structure of

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8 An investigation of Huygens’s correction of these rules would be an extension of the project of tracing the conceptual presuppositions of *Principles* II.1-18 into the explanations of natural phenomena found in the rest that work.
formalization, which he showed to have its origin in the invention of symbolic algebra. Taking
the works of Klein and Husserl as its foundation, the present study has attempted to further the
project of desedimentation by fleshing out and building upon Klein’s analysis of Descartes’s
contribution to the development of modern mathematics, and then using that as a basis upon
which to unpack Klein’s brief comments about the concept of space. In doing so, Klein’s
account of the genesis of modernity was extended beyond the mathematical realm, in which
modern conceptuality first arose. By contributing to Klein’s overall account of the development
of modern thought, I hope to have corroborated the validity and importance of his work, while
also providing a starting-point for tracing the further conceptual developments of modern
mathematical physics. For, on Klein’s account, it is only through the slow and careful process of
tracing such developments that we can achieve a full understanding of the way modern
mathematical science has shaped modernity at large.
Primary Sources


Secondary Sources


