THE CATHOLIC UNIVERSITY OF AMERICA

Full-waveform Inversion of Seismic Input Motions in a Near-surface Domain

A DISSERTATION

Submitted to the Faculty of the
Department of Civil and Environmental Engineering
School of Engineering
Of The Catholic University of America
In Partial Fulfillment of the Requirements
For the Degree
Doctor of Philosophy

©

Copyright
All Rights Reserved
By
Bruno Peruqui Guidio

Washington, D.C

2020
Full-waveform Inversion of Seismic Input Motions in a Near-surface Domain

Bruno Peruqui Guidio, Ph.D.

Directed by: Chanseok Jeong, Ph.D.

This research aims to develop a new inverse modeling approach to allow engineers to identify the spatial and temporal distribution of unknown seismic input motion (i.e., incoming seismic wavefield) in a near-surface domain by using sparsely-available measurement data of vibrational motions in the domain.

Because existing methods are limited due to complexity, inaccuracies, and uncertainties, this research explores a new computational method for identifying seismic input motion.

The main body of this dissertation is composed of three main Chapters. First, Chapter 2 considers out-of-plane (henceforth, SH) wave motions in a two-dimensional (2D) bounded domain. The partial-differential-equation (PDE)-constrained optimization framework is employed to search a set of control parameters, by which a misfit between measured responses at sensors on the top surface induced by targeted traction and their computed counterparts induced by estimated traction is minimized. To mitigate the solution multiplicity of the presented inverse problem, we employ the Tikhonov (TN) regularization on the estimated traction function and test its performance. We present the mathematical modeling and numerical implementation of both optimize-then-discretize (OTD) and discretize-then-optimize (DTO) approaches. We also compare their performances each other. The finite element method (FEM) is employed to obtain the numerical solutions of state and adjoint problems. Newton’s method is utilized for estimating an optimal step length in combination with the conjugate-gradient scheme, calculating the desired search direction, throughout a minimization process. The numerical results in Chapter 2 present that, first, the complexity of a
material profile in a domain increases the error between reconstructed traction and its target. Second, the OTD and DTO approaches lead to the same inversion result. Third, when the sampling rate of the measurement is equal to the timestep for discretizing estimated traction, the ratio of the size of measurement data to the number of the control parameters can be as small as 1:12 in the presented work. Fourth, it is acceptable to tackle the presented inverse modeling of dynamic traction without the TN regularization. Fifth, the inversion performance is more compromised when the noise of a more significant level is added to the measurement data, and using the TN regularization does not improve the inversion performance when noise is added to the measurement. Sixth, our minimizer suffers from solution multiplicity less when it identifies dynamic traction of lower frequency content than that of higher frequency content.

Second, Chapter 3 presents a new inversion modeling method for reconstructing complex, incoherent seismic input motions in a domain that is truncated by a wave-absorbing boundary condition (WABC). In a set of numerical examples, targeted dynamic traction at the WABC mimics seismic incident wavefield, and FEM is used to solve state and adjoint problem. In Chapter 3, we use only the DTO approach without any regularization because the numerical examples in Chapter 2 shows that the DTO approach performs as successfully as the OTD counterpart, and the TN regularization is not necessary. The numerical results in Chapter 3 show that the incident, inclined plane waves, cannot be fully reconstructed by using the sensors only on the top surface. It is necessary to include a vertical array of sensors on the side boundary of a domain. Second, the minimizer suffers more from solution multiplicity when it identifies traction of a higher dominant frequency. Third, the inversion solver can reconstruct traction of a complex, realistic seismic signal. Lastly, the error between the reconstructed traction and its counterpart is slightly higher in a two-layered background domain with inclusions than in a single-layered counterpart with inclusions.
In Chapters 2 and 3 of the presented work, the material proprieties of the domain are assumed to be known in advance. This assumption does not always hold in realistic settings. Chapter 4 is focused on investigating the feasibility of resolving this challenge, in structural health monitoring problem settings where the proprieties are not known, by using a new joint inversion perspective. Accordingly, Chapter 4 presents a computational study to investigate the feasibility of simultaneous identification of a material property of a Timoshenko continuous beam and a moving vibration source on the beam by using the data of measured vibrations. The finite element method is employed to solve the wave equations of a Timoshenko beam subject to a moving vibrational source. It uses the Genetic Algorithm (GA) as an inversion solver to identify the values of targeted control parameters that characterize the material property of the beam and a moving vibration source. The numerical results in Chapter 4 show that the presented inversion method can detect the characteristics of a moving wave source as well as the spatial variation of the elastic modulus of a Timoshenko-beam continuous bridge model, which is set to be piece wisely homogeneous in this computational study.

The contributions of the presented dissertation are as follows. Our newly-developed adjoint equation-based inverse-source procedure can fully reconstruct seismic traction inputs propagating into a multi-dimensional, bounded heterogeneous soil domain and in a domain truncated by using WABC. We also found that it is feasible to identify the characteristics of a moving wave source as well as the spatial variation of the elastic modulus of a Timoshenko-beam continuous bridge model by using measured vibration data induced by a random moving wave source (e.g., a vehicle).
This thesis by Bruno Peruqui Guidio fulfills the thesis requirement for the doctoral degree in Civil and Environmental Engineering approved by Chanseok Jeong, Ph.D., as Director, and by Diego Turo, Ph.D., Jandro Abot, Ph.D., and Laura Micheli, Ph.D. as Readers.

Chanseok Jeong, Ph.D., Director

Diego Turo, Ph.D., Reader

Jandro Abot, Ph.D., Reader

Laura Micheli, Ph.D., Reader
To my wife, son, parents and brother for all of their love and support.
"For from him and through him and for him are all things. To him be glory forever. Amen" (Rm 11, 36)
# TABLE OF CONTENTS

- **LIST OF FIGURES** ................................................................. ix
- **LIST OF TABLES** ................................................................. xiv
- **ACKNOWLEDGMENTS** ............................................................ xv

## CHAPTER

1. **INTRODUCTION** ............................................................... 1
   1.1 Literature Review .......................................................... 1
   1.2 Problem Statement and Proposed Solutions.......................... 7
   1.3 Dissertation Contribution .............................................. 9
   1.4 Dissertation Outline .................................................. 9

2. **FULL-WAVEFORM INVERSION OF INCOHERENT DYNAMIC TRACTION IN A BOUNDED 2D DOMAIN OF SCALAR WAVE MOTIONS.** .................. 11
   2.1 Problem Definition .................................................... 11
      2.1.1 The governing equation ........................................ 11
      2.1.2 Parameterization of an estimated dynamic traction function .... 13
   2.2 Inverse Modeling—the optimize-then-discretize (OTD) approach .......... 14
      2.2.1 The objective functional ....................................... 14
      2.2.2 Lagrangian functional .......................................... 16
      2.2.3 The first-order optimality conditions ....................... 16
      2.2.4 Finite element solution of the state problem ................ 23
      2.2.5 Finite element solution of the adjoint problem ............ 24
2.2.6 Time integration ................................................................. 25
2.2.7 The discrete form of the gradient ....................................... 26

2.3 Inverse Modeling—the discretize–then-optimize (DTO) approach .......... 26

2.3.1 The discrete objective functional ....................................... 26
2.3.2 The discrete Lagrangian functional .................................... 27
2.3.3 The first-order optimality condition in the DTO modeling .......... 28
2.3.4 Implementation of the regularization term in the gradient ............ 30

2.4 Numerical Implementation of the Inversion Process .......................... 33

2.4.1 Conjugate gradient ......................................................... 34
2.4.2 Adaptively-calculated regularization factor ............................. 35
2.4.3 Updating the estimated control parameters .............................. 36

2.5 Numerical Experiments ......................................................... 37

2.5.1 Example 1: Investigating the inversion performance with respect to the material profile complexity .................................................. 42
2.5.2 Example 2: Comparison between the inversion performances of the OTD and DTO approaches ................................................... 47
2.5.3 Example 3: Investigating the inversion performance with respect to the number of sensors .................................................... 47
2.5.4 Example 4: Investigating the inversion performance with respect to \( I_R \) ............................................................... 50
2.5.5 Example 5: Investigating the inversion performance with respect to the noise level ............................................................ 52
2.5.6 Example 6: Examining the feasibility of the presented inverse modeling to reconstruct a realistic seismic signal \( F_2(x,t) \) .................. 54

2.6 Summary .................................................................................. 60

3. FULL-WAVEFORM INVERSION OF SEISMIC INPUT MOTIONS IN A DOMAIN TRUNCATED BY USING ABSORBING BOUNDARY CONDITIONS ......................................................... 62

3.1 Problem Definition ............................................................... 62

3.1.1 The governing equation .......................................................... 63

3.2 Forward Wave Modeling .......................................................... 64
3.3 Inverse Modeling ................................................................. 67
  3.3.1 Parameterization of a dynamic input traction function ............... 67
  3.3.2 Discrete objective and Lagrangian functional .......................... 68
  3.3.3 The first order optimality conditions .................................. 68

3.4 Numerical Experiments ......................................................... 71
  3.4.1 The exemplary forward wave responses .................................. 75
  3.4.2 Demonstration of the relation between the traction and $u^I$ on a WABC boundary ................................................................. 78
  3.4.3 The verification of the inverse modeling .................................. 78
  3.4.4 Example 1: Investigating the inversion performance with/without a vertical array of sensors .................................................. 79
  3.4.5 Example 2: Investigating the inversion performance with respect to the dominant frequency of $P(\gamma, t)$ ........................................ 82
  3.4.6 Example 3: Examining the feasibility of the presented inverse modeling to reconstruct a realistic seismic signal $P_d(\gamma, t)$ ............ 82
  3.4.7 Example 4: Investigating the inversion performance in a 2-layered background domain ......................................................... 88

3.5 Summary ............................................................................. 94

4. SIMPLE JOINT INVERSION USING GENETIC ALGORITHM (GA) .......... 96
  4.1 Problem Definition ............................................................... 96
  4.2 Forward Wave Modeling ......................................................... 99
    4.2.1 Finite Element Method ...................................................... 99
    4.2.2 Verification of the forward wave modeling ............................. 101
  4.3 Inverse Modeling ................................................................. 103
  4.4 Numerical Experiments .......................................................... 104
    4.4.1 Example 1 (Cases 1 to 5): joint inversion of two source parameters and three stiffness parameters in a bridge comprised of three piece wisely-homogeneous segments ........................................... 105
    4.4.2 Example 2 (Cases 6 to 9): joint inversion of two source parameters and nine stiffness parameters in a bridge comprised of nine piece wisely-homogeneous segments .............................................. 114
4.5 Summary ................................................................. 117

5. CONCLUSIONS .......................................................... 118

5.1 Summary of the dissertation ......................................... 118
5.2 Future extensions ...................................................... 119

BIBLIOGRAPHY .............................................................. 121
LIST OF FIGURES

<table>
<thead>
<tr>
<th>Figure</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.1</td>
<td>Proposed solution: Target the reconstruction of effective seismic input motions in a near-surface truncated solid.</td>
</tr>
<tr>
<td>2.1</td>
<td>Problem setting of Chapter 2.</td>
</tr>
<tr>
<td>2.2</td>
<td>$F_{kj}$ is surrounded by four elements in the space in terms of $x$ and $t$: the horizontal and vertical axes represent, respectively, the $x$ coordinate and time $t$.</td>
</tr>
<tr>
<td>2.3</td>
<td>(a) The time signal of $F_1(x = 30, t)$; (b) the amplitude of Fourier Transform of $F_1(x = 30, t)$; (c) the time signal of $F_2(x = 30, t)$; and (d) the FFT of $F_2(x = 30, t)$.</td>
</tr>
<tr>
<td>2.4</td>
<td>Heterogeneous solids: (a) Material profile 2; and (b) Material profile 3.</td>
</tr>
<tr>
<td>2.5</td>
<td>Comparison between the gradients generated by the OTD approach, DTO approach and the FD approximation.</td>
</tr>
<tr>
<td>2.6</td>
<td>(a) Target and (b-d) Reconstructed $F_1(x, t)$, in N/m², for Cases 1-3 at the 1000th iteration. Horizontal and vertical axis in the contour plot represent the numbering of the discretized points over space ($x$) and time ($t$) of the distribution of $F_1(x, t)$, respectively.</td>
</tr>
<tr>
<td>2.7</td>
<td>Example 1 - Objective function, $L$, versus iterations with respect to the material profiles.</td>
</tr>
<tr>
<td>2.8</td>
<td>Example 1 - After 1000 iterations, the final value of $\mathcal{E}$ is 11.04% in Case 1, 16.87% in Case 2, and 25.02% in Case 3.</td>
</tr>
</tbody>
</table>
2.9  \(u_m\) and \(u\), at sensors placed on the top surface. (a-b) measured at \(x = 20\) m and \(x = 40\) m, respectively, for Case 1. (c-d) measured at \(x = 20\) m and \(x = 40\) m, respectively, for Case 2. (e-f) measured at \(x = 10\) m and \(x = 20\) m, respectively, for Case 3.

2.10 Wave responses, \(u(x, y, t)[m]\), at 1.5 seconds due to (a) target traction and (b) reconstructed traction in Case 3, and (c) the difference between them.

2.11 Wave responses, \(u(x, y, t)[m]\), at 1.5 seconds due to (a) target traction and (b,c) reconstructed traction for the OTD and DTO approach, respectively. (d,e) The difference in wave responses between target and OTD approach and target and DTO approach, respectively.

2.12 Example 2 - \(L\) and \(E\) with respect to the inverse approach.

2.13 Example 3 - Error, \(E\), versus iterations with respect to the number of sensors.

2.14 Example 4 - Objective functional, \(L\), versus iterations with respect to \(I_R\).

2.15 Example 4 - Error, \(E\), versus iterations with respect to \(I_R\).

2.16 Example 4 - Estimated traction \(F_1(x, t)\), in N/m\(^2\), for (a) Case 8, where \(I_R = 1.0\), and (b) Case 11, where \(I_R = 0.0\).

2.17 Example 5 - \(L\) and \(E\) with respect to the noise level.

2.18 Example 5 - \(L\) and \(E\) for 2% of noise with respect of \(I_R\).

2.19 Example 6 - (a) Target and (b) Reconstructed \(F_2(x, t)\), in N/m\(^2\), for Case 18 at the 6000-th iteration.

2.20 Example 6 - Objective function, \(L\), decreases without the sawtooth behavior because \(I_R\) of 0.0 is used.

2.21 Example 6 - After 6000 iterations, the final value of \(E\) is 3.85% in Case 18.

2.22 Wave responses, \(u(x, y, t)[m]\), at 6.0 seconds due to (a) target load and (b) reconstructed load in Case 18, and (c) the difference between them.

3.1 The problem configuration using WABC.
3.2 Material profile 2: a 2-layered solid with two inclusions. ............................... 72

3.3 (a,c,e,g) The time signal of \( P_{1,2,3,4}(\gamma = 160,t) \), respectively; (b,d,f,h) the amplitude of Fourier Transform of \( P_{1,2,3,4}(\gamma = 160,t) \), respectively. ............................... 73

3.4 (a-d) The wave motions of \( u(x,y,t)[m] \), at \( t = 0.3 \) s, \( 0.4 \) s, \( 0.5 \) s, and \( 0.6 \) s, respectively, in Case 1. ............................... 76

3.5 (a-d) The wave motions of \( u(x,y,t)[m] \), at \( t = 0.3 \) s, \( 0.4 \) s, \( 0.5 \) s, and \( 0.6 \) s, respectively, in Case 6. ............................... 77

3.6 (a) Input traction \( P_{1}(\gamma,t) \) and (b) \( \frac{2G}{V} \frac{\partial u}{\partial t} \), in N/m². ............................... 78

3.7 Comparison between the gradients generated by the presented inverse modeling and the FD approximation ............................... 79

3.8 Example 1: (a) Target and (b,c) Reconstructed \( P_{1}(\gamma,t) \) in N/m², for Case 1 and 2 at the 1000th iteration. Horizontal and vertical axis in the contour plot represent, respectively, the numbering of the discretized points over space (\( \gamma \)) and time (\( t \)) of the distribution of \( P_{1}(\gamma,t) \). ............................... 80

3.9 Example 1 - After 1000 iterations, the final value of \( \mathcal{E} \) is 9.39% in Case 1, and that in Case 2 is 1.72%, which shows a reduction of 7.67% in \( \mathcal{E} \). ............................... 81

3.10 Example 2: (a) Target and (b,c) Reconstructed \( P_{1}(\gamma,t) \) in N/m²; (c) Target and (d) Reconstructed \( P_{2}(\gamma,t) \) in N/m²; and (e) Target and (f) Reconstructed \( P_{3}(\gamma,t) \) in N/m², for Cases 2-4 at the 1000th iteration. ............................... 83

3.11 Example 2 - After 2000 iterations, the terminal value of \( \mathcal{E} \), 1.32%, for reconstructing \( P_{1}(\gamma,t) \) in Case 2 is smaller than its counterparts, 3.38%, of rebuilding \( P_{2}(\gamma,t) \) in Case 3 and 11.02% for reconstructing \( P_{3}(\gamma,t) \) in Case 4. ............................... 84

3.12 Example 2 - Case 2: Wave responses, \( u(x,y,t)[m] \), in the domain induced by (left) the targeted \( P_{1}(\gamma,t) \) and (middle) its reconstructed counterpart, and (right) the difference between them at (a-c) 0.3 s, (d-f) 0.4 s, (g-i) 0.5 s, and (j-l) 0.6 s. ............................... 85
3.13 Example 2 - Case 3: Wave responses, \(u(x, y, t)[m]\), in the domain induced by (left) the targeted \(P_2(\gamma, t)\) and (middle) its reconstructed counterpart, and (right) the difference between them at (a-c) 0.3 s, (d-f) 0.4 s, (g-i) 0.5 s, and (j-l) 0.6 s.

3.14 Example 2 - Case 4: Wave responses, \(u(x, y, t)[m]\), in the domain induced by (left) the targeted \(P_3(\gamma, t)\) and (middle) its reconstructed counterpart, and (right) the difference between them at (a-c) 0.3 s, (d-f) 0.4 s, (g-i) 0.5 s, and (j-l) 0.6 s.

3.15 Example 3: (a) Target and (b) Reconstructed \(P_4(\gamma, t)\), in N/m^2, for Case 5 at the 1000th iteration.

3.16 Example 3 - \(L\) and \(E\) comparing Case 2 and Case 5.

3.17 Example 3: Wave responses, \(u(x, y, t)[m]\), in the domain induced by (left) the targeted \(P_4(\gamma, t)\) and (right) its reconstructed counterpart at (a-b) 1.0 s, (c-d) 2.5 s, (e-f) 3.0 s, and (g-h) 3.5 s.

3.18 Example 3: Wave responses, \(u(x, y, t)[m]\), in the domain induced by (left) the targeted \(P_4(\gamma, t)\) and (right) its reconstructed counterpart at (a-b) 3.75 s, (c-d) 4.5 s, (e-f) 5.25 s, and (g-h) 6.0 s.

3.19 Example 4: (a) Target and (b) Reconstructed \(P_5(\gamma, t)\), in N/m^2, for Case 6 at the 1000th iteration.

3.20 Example 4 - The final value of \(E\), in Case 6, is 1.91%, and that in Case 2 is 1.72%.

3.21 \(u_m\) and \(u\) at sensors (a) on the top surface measured at \(x = 99\) m and (b) on the right boundary measure at \(\gamma = 288\) m for Case 6.

3.22 Example 4: Wave responses, \(u(x, y, t)[m]\), in the domain induced by (left) the targeted \(P_5(\gamma, t)\) and (middle) its reconstructed counterpart, and (right) the difference between them at (a-c) 0.3 s, (d-f) 0.4 s, (g-i) 0.5 s, and (j-l) 0.6 s.

4.1 Problem configuration of Chapter 4.

4.2 Two snapshots of \(H(x, t)\) at \(t = 0\) and 1 s using \(x_0 = 50\) m and \(\vartheta = 20\) m/s.
4.3 Comparison between \( u(5,t) \) generated by our FEM wave solver and that by a reference code for a simply-supported beam of its length 10 m subject to a uniformly-distributed sinusoidal loading of its frequency 2 Hz.

4.4 A piece wisely-homogeneous Timoshenko beam with three segments in Example 1.

4.5 Snapshots of the exemplary wave response, \( u(x,t) \), and the targeted source function, \( q(x,t) \), in Example 1 considering the targeted material profile.

4.6 The Doppler effect of the wave responses induced by a moving source in Example 1 considering the targeted material profile.

4.7 The average error for the best-fit individual versus the GA iterations in Cases 2, 3, 4 and 5 of Example 1.

4.8 The misfit functional for the best-fit individual over the GA iterations in Case 1.

4.9 The histograms of (a) \( P \) and (b) \( f \) of the entire individuals at all the generations in Case 1.

4.10 The histograms of (a) \( E_1 \), (b) \( E_2 \), and (c) \( E_3 \) of the entire individuals at all the generations in Case 1.

4.11 Wave responses, \( u^m \) and \( u \), at the sensors in Case 1.

4.12 A piece wisely-homogeneous Timoshenko beam with nine segments in Example 2.

4.13 The reconstructed elastic modulus of a piece wisely-homogeneous beam of nine segments via the joint inversion in Example 2: (a) Case 6 using PS of 50, (b) Case 7 using PS of 100, (c) Case 8 using PS of 200, and (d) Case 9 using PS of 400.

4.14 (a) \( \mathcal{L} \) and (b) \( \mathcal{E} \) for the best-fit individual versus the GA iteration in Example 2.
# LIST OF TABLES

<table>
<thead>
<tr>
<th>Table</th>
<th>Description</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>2.1</td>
<td>Summary of all cases in Chapter 2.</td>
<td>40</td>
</tr>
<tr>
<td>3.1</td>
<td>Summary of all cases in Chapter 3.</td>
<td>74</td>
</tr>
<tr>
<td>4.1</td>
<td>Example 1: joint inversion of two source parameters and three stiffness</td>
<td>108</td>
</tr>
<tr>
<td></td>
<td>parameters in a bridge comprised of three piece wisely-homogeneous segments</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(GN = 50 for all the cases 1 to 5).</td>
<td></td>
</tr>
<tr>
<td>4.2</td>
<td>Example 2 - joint inversion of two source parameters and nine stiffness</td>
<td>117</td>
</tr>
<tr>
<td></td>
<td>parameters in a bridge comprised of nine piece wisely-homogeneous segments</td>
<td></td>
</tr>
<tr>
<td></td>
<td>by using NS of 45, GN of 50, and PS of different values: while only the</td>
<td></td>
</tr>
<tr>
<td></td>
<td>source parameters are shown in this table, the stiffness parameters are</td>
<td></td>
</tr>
<tr>
<td></td>
<td>visualized in Fig. 4.13.</td>
<td></td>
</tr>
</tbody>
</table>
ACKNOWLEDGMENTS

First of all, I want to give my sincerest gratitude to Dr. Chanseok Jeong for the great patience, understanding, and guidance he shared with me upon the completion of this dissertation. I really appreciate his valuable advice and care along the way. I am also grateful to him for sharing his enlightenment on many topics on theoretical and computational aspects of this research, without his optimism and belief in my skill, I would not be able to have the motivation to finish this research.

I would also like to express my appreciation for my dissertation committee member, Dr. Diego Turo, Dr. Jandro Abot, and Dr. Laura Micheli, for reviewing my work and providing insightful comments on this dissertation. Additionally, I would like to thank every person in the Engineering Department that helped me while at Catholic University. All the professors, staff, and friends, my deepest gratitude for the past years in this life-changing graduate studies.

I want to thank my wife, Driele Guidio. Without her support and love, I would not be able to attend the Catholic University of America and fulfill my dream of completing a doctoral degree. Also, without her, I would not be able to have one of the happiest moments in my life, the life of our first son, Thomas Guidio. Thank you so much, amor, for everything that we have built so far. I love you and our son.

I would like to thank my family and friends for giving me words of encouragement. Extraordinarily, I would like to express my appreciation to my father, mother, and brother for giving me the inspiration to do my best in everything I do.
Finally, I want to give thanks to God for taking care of my family and me. Thank you, God, for allowing me to obtain my degree at CUA.

My graduate studies were supported by the CUA School of Engineering and the National Science Foundation (NSF)—NSF grant (CMMI-1855406) ‘Full-waveform inversion of seismic input motions in a truncated domain’. I am grateful to these supports very much.
CHAPTER 1
INTRODUCTION

1.1 Literature Review

There is a need for estimating incident seismic wavefields in a soil-structure system from limited seismic measurement data because, by using the identified seismic inputs, engineers can reconstruct (i.e., replay) responses within structures and soils during an earthquake event and study the effect of an earthquake on the built environment, including subsurface systems (soil, foundations, and underground structures). There have been two dominant, conventional methods for the purpose mentioned above: the one is deconvolution and the other is the inversion of a seismic source profile at a fault in a very large domain.

The deconvolution algorithm has been used for the identification of an incoming seismic wave signal into a soil column by using vibrational measurement on the ground surface. For instance, Mejia and Dawson [1] have presented the deconvolution to compute a seismic input signal by using the SHAKE program [2], which solves the 1D seismic wave propagation problem in a domain of a semi-infinite extent. Recently, there have been studies on the deconvolution of both vertical and horizontal components of surficial measurement data to identify the vertical and horizontal input wave motions [3, 4]. We note that, although the deconvolution has been widely used in geotechnical earthquake engineering, it is effective on individual soil columns only when soil stratification is horizontally uniform, and incoming seismic waves vertically propagate. Namely, when the soil property is arbitrarily heterogeneous (not horizontally layered), and incoming seismic waves,
consisted of P, S, and/or surface waves, are highly incoherent (not vertically propagating), the deconvolution cannot effectively reconstruct the incoming waves.

On the other hand, there have been studies for inverting for seismic-source parameters (however simple or complicated an adopted seismic source model may be) at a hypocenter. For instance, Akcelik et al. [5] presented a method to invert for a simplified seismic source time signal in a large 3D domain that includes a source at a fault. This method requires forward and inverse wave simulations of a very large domain. Upon characterizing the seismic source via inversion, then attention is typically turned on how to propagate the motion from the source to the surface, where the real interest is. However, in the large-scale source-inversion problem, the material properties of a large domain could be poorly characterized (on the other hand, the material properties could be better characterized by virtue of active wave source-based geotechnical characterization method [6]).

The limitations of the two methods, mentioned above, necessitate developing an alternative method that can identify arbitrary, incoherent incoming seismic waves in a truncated 2D or 3D domain by using sparse seismic measurement. Such a potential method could serve as an alternative to the deconvolution, while bypassing all the complexities associated with the inversion of the source at the hypocenter and the subsequent propagation steps. Recently, Jeong and Seylabi [7] presented prototype research that can reconstruct a seismic input signal propagating into a 1D truncated, heterogeneous, undamped solid system by using the partial differential equation (PDE)-constrained optimization method. Lloyd and Jeong [8] also show that the PDE-constrained optimization can reconstruct the discretized parameters of moving vibrational body forces in both space and time in a 1D heterogeneous, linear, elastic, undamped solid by using the sparse measurement of wave motions. These works were cast into a minimization problem, where a misfit between a measured response(s) at a sensor(s) induced by a targeted wave source profile and a computed wave solution(s) induced by an estimated source profile is minimized, and the PDE-constrained optimization scheme analytically evaluates the gradient of a misfit with respect to control parameters,
which parameterize an estimated dynamic input function. Because of such an analytical nature, its computational efficiency of computing the gradient of a misfit with respect to control parameters does not depend on the number of them. Thus, it can update a large set of control parameters very efficiently. In addition, the PDE-constrained optimization can accommodate regularization to address the solution multiplicity of full-waveform inversion problems, typically caused by the sparsity of measurement, and to stabilize the convergence by penalizing an undesired aspect of an estimated profile while enhancing a selected feature (e.g., smoothness) of targeted profiles.

There had been a wide range of studies on elastodynamic inverse problems—e.g., full-waveform material inversion, inverse material design, full-waveform inverse-scattering, and source inversion—based on the PDE-constrained optimization as shown in the following literature review. Kang and Kallivokas [9, 10], examined the numerical algorithms to image the distributions of the scalar-wave speeds in one-dimensional and two-dimensional solids that are surrounded by Perfectly-Matched-Layers (PML), where waves are forced to decay and they are prevented from reflecting off the surrounding boundaries [11, 12, 13, 14, 15]. Pakravan et al. [16] devised a new methodology to probe the elastic and attenuating parameters of two-dimensional viscoelastic layered solids. Kallivokas et al. [17], Fathi et al. [18, 6], and Kucukcoban et al. [19] studied algorithms to invert for the Lamé parameters in two-dimensional and three-dimensional PML-truncated solid domains, and these computational studies made significant advancement in geotechnical site characterization using dynamic tests. Tran and McVay [20] investigated the Gauss-Newton-based full waveform inversion approach to estimate the elastic modulus profile in a two-dimensional domain. Mashayekh et al. [21] investigated a new methodology to estimate the mechanical properties of layered elastic or viscoelastic media by taking into account the dispersion relation of the layered medium. Tromp et al. [22] and Zhu et al. [23] investigated the adjoint-tomography geophysical inversion using the spectral element wave modeling in global- or regional-scale domains by using earthquake waves emitted from a seismic source of a known location and a known signal. Recently, Goh and Kallivokas [24] investigated a new inverse metamaterial design method by min-
imizing the distance between the target and the computed group velocity profiles via a dispersion-constrained optimization method, and the method can be used for designing metamaterials, such as user-defined omnidirectional band gaps in an elastic medium. In addition, it had been shown that strong discontinuities within solids, such as the boundaries of voids, can be identified by using inverse modelings. Namely, Guzina et al. [25] and Jeong et al. [26] studied inverse scattering algorithms using the PDE-constrained optimization, associated with the boundary element method (BEM) wave solver, taking advantages of the moving boundary concept and the total derivative [27]. Nguyen-Tuan [28] also made recent progress in the inverse scattering algorithm, using the PDE-constrained optimization associated with the level-set based extended finite element method (XFEM) solver, so as to identify the geometry of voids in a static-hydro-mechanical system. Both BEM and XFEM wave solvers can model the boundaries of the strong discontinuities and update their geometries without cumbersome remeshing during an inversion process as opposed to a conventional finite element method (FEM) wave solver, which should remesh a domain to update the boundaries’ geometries [29]. On the other hand, Aquino et al. [30] devised a novel algorithm to detect debonded interfaces (i.e., interface cracks or incomplete weld bonds) in composite solids by using steady-state vibrational tests and a density function that characterizes the bonding at the interfaces in composite solids. Besides, the following studies have investigated the methods to identify dynamic input functions. Hasanov and Baysal [31] studied an algorithm to detect the time-independent spatial load distributions of a dynamic source on a cantilever beam. Binder et al. [32] also recover virtual, stationary wave sources at possible locations of structural anomalies using the adjoint equation approach. Walsh et al. [33] reported the inverse problems for the identification of dynamic sources in acoustics and elastodynamics, employing a DTO approach. That is, the discrete form of the forward wave equation at each time step is imposed into a Lagrangian, and the adjoint equation and the gradient of the Lagrangian with respect to the source parameters are derived in discrete forms. In particular, Walsh et al. [33] suggested that the DTO approach is more suitable than the OTD counterpart when the nonlinearity is considered in the
forward problem because the discrete, linearized forward equation of every time step can be individually side-imposed into the Lagrangian. The PDE-constrained optimization has been also used for identifying optimal, non-moving wave source profiles that can focus wave energy to specific areas in solids [34, 35, 36, 37, 38, 39, 40].

Due to the aforementioned robustness and scalability of the PDE-constrained optimization for inverse problems, it is worth continuing to investigate it so as to identify incoherent seismic input motions in a multi-dimensional domain from sparse seismic measurement data. This dissertation reconstructs the spatial and temporal distributions of incoherent dynamic traction on a boundary of a heterogeneous, bounded, undamped solid system of anti-plane motions by using the PDE-constrained optimization method.

Also, there is a need to characterize the spatial distributions of the material properties of transportation infrastructures (e.g., a bridge, a tunnel, a roadway, and a railway) and find any anomaly of their material properties (e.g., reduced stiffness caused by corrosion or cracks in their structural members). To this end, engineers employ vibration-based structural health monitoring (SHM) methods by employing active wave sources (e.g., impacts or vibrations) of known signals onto an inspected structure and measuring corresponding vibration responses on it [41]. From those measured vibrations, engineers back-calculate the properties of the structure [42]. There have been recent development of theoretical and computational studies for identifying the material properties of a solid structure by using sparsely-measured vibration data induced by active wave sources.

Despite the aforementioned extensive development on the full-waveform inversion algorithms, as the disadvantages of an active wave source-based SHM approach, it should disrupt traffics on or in the vicinity of an inspected transportation structure—the traffics should be stopped to minimize random noises in measurement data—, and it is costly to apply the approach to infrastructures frequently in densely-populated areas. Therefore, there is a need to develop an alternative to an active wave source-based SHM approach.
In order to seek such an alternative method, this research studies a passive wave source-based SHM approach, by which engineers can take advantage of ambient vibration sources, such as vehicles on roadway or trains on railroads. As an advantage of the passive wave source-based SHM approach, elastic waves, induced by strong vibrational forces (e.g., tractions exerted by moving trailer trucks or trains on an inspected structure), can reach far fields of the structure—including not only the structure but also the soils and foundations under the structure. Thus, the measured data of waves can carry information about the mechanical properties of infrastructures of large extents. As another advantage of the passive wave source-based SHM approach, engineers can take advantage of unlimited amounts of measured data from a network of modern, ubiquitous sensors, e.g., fiber optic cables [43], in infrastructures. Because engineers can measure the traffic-induced ambient vibrations on an inspected infrastructure on a day-to-day basis without interrupting its normal operations, they can identify its material properties frequently. As related studies to the passive wave source-based material characterization, Akcelik et al. [5] simultaneously inverted for a simplified seismic source time signal and material properties in a large 3D truncated domain that includes a stationary seismic source by using the full-waveform inversion method. Cavadas et al. [44] studied a pattern recognition method and examined vibration data due to regular traffic, to detect wave sources’ information and the location of stiffness reduction in a beam, considering only the quasi-static components of vibration responses (instead of their full time-domain waveform signals). Liu et al. [45] investigated signal processing and dimensionality-reduction techniques that can identify the relationship between the damage severity of a structure and the vibration responses of a passing vehicle by using mobile sensors installed on the vehicle to determine the location of the damage. Despite the recent development mentioned above, we note that, to date, the literature is not mature on the theoretical and computational studies to back-calculate the spatial distributions of material properties of infrastructures by using traffic-induced ambient vibration signals.
To fill this gap, this dissertation, specifically chapter 4, attempts to investigate the feasibility of the joint inversion to identify unknown information of the stiffness of a 1D beam bridge model and a moving wave source on it. When an inverse problem is aimed at identifying the distributions of multiple independent variables (e.g., the simultaneous inversion of the stiffness and Poisson’s ratio of a solid [46]), it is known that the inverse problem suffers from solution multiplicity more severely than that aimed at identifying a single variable. To address the solution multiplicity of the presented joint inversion problem, the Genetic Algorithm (GA) is employed, which is known to be a global optimization method for an optimization problem of a small number of control parameters. Here, as a structural model, I consider the Timoshenko beam model instead of the Euler-Bernoulli model, which does not take into account the effects of shear deformation and rotational inertia of a beam. Namely, this research considers the Timoshenko beam theory, considering a beam’s shear deformation and its rotational inertia, without which the accuracy of wave responses of a beam is compromised [47].

1.2 Problem Statement and Proposed Solutions

To date, there has been no robust numerical method that can identify complex, incoherent seismic input motions in a near-surface domain, truncated by a wave-absorbing boundary condition (WABC). Existing methods are limited to either deconvolution, which can reconstruct only a vertically-propagating, coherent incident wave in a horizontally layered soil column, or large-scale seismic-source inversion that can identify seismic source parameters at a hypocenter. However, there are many complexities and uncertainties associated with the large-scale inversion that render it impractical for near-surface simulations. This dissertation seeks to bypass the complexities associated with the large-scale seismic source inversion by targeting the reconstruction of effective seismic input motions at a truncation boundary in a near-surface domain, using a partial differential equation (PDE)-constrained optimization method.
In other words, this dissertation investigates a new method to identify seismic incident wave-field that propagates into a near-surface domain of interest from sparse vibrational measurement data. By using the identified seismic inputs, engineers could potentially reconstruct (i.e., replay) responses within structures and soils during an earthquake event (see Fig. 1.1).

Figure 1.1: Proposed solution: Target the reconstruction of effective seismic input motions in a near-surface truncated solid

Therefore, if successfully developed, this method can pinpoint where large amplitudes of stress waves occur (or structural failures occur) in built environments in a domain of interest during seismic events.
1.3 Dissertation Contribution

This dissertation leads to an unbiased, accurate approach for identifying seismic input motions in a truncated solid system. The method can identify the comprehensive profiles (i.e., both spatial and temporal distributions) of targeted seismic-input wave fields without any prior information about them. In addition, the method will automatically and accurately take into account wave reflections and refractions, which are induced by complex geometries and heterogeneities in a domain. It is because that the FEM wave solvers can accurately model such complexities in the state and adjoint problems.

• Chapter 2 investigates the mathematical and computational modeling of the new seismic-input identification method to identify incoherent dynamic traction in a multi-dimensional domain for the first time.

• Chapter 3 presents a robust numerical method that can identify complex, incoherent seismic incident wavefield in a solid, truncated by a WABC for the first time.

• Chapter 4 shows that it is feasible to conduct the simultaneous identification of both the material property of a Timoshenko beam and a moving vibration source on the beam by using the data of measured vibrations on the beam for the first time. Based on the feasibility study in Chapter 4, the presented seismic-input inversion method could be extended such that it can be effective even in a domain, of which material properties are not known in advance.

1.4 Dissertation Outline

The content of this dissertation is organized as follows. In Chapter 2, an adjoint equation-based inverse-source procedure is investigated to reconstruct the spatial and temporal distribution of incoherent dynamic traction in a bounded domain. This work considers anti-plane wave motions in a two-dimensional (2D) domain. The PDE-constrained optimization framework is employed to search a set of control parameters, by which a misfit between measured responses at
sensors on the top surface induced by targeted traction and their computed counterparts induced by estimated traction is minimized. The mathematical modeling and numerical implementation of both optimize-then-discretize (OTD) and discretize-then-optimize (DTO) approaches are presented. The finite element method (FEM) is employed to obtain the numerical solutions of state and adjoint problems. Newton’s method is utilized for estimating an optimal step length in combination with the conjugate-gradient scheme, calculating a desired search direction, throughout a minimization process. Numerical experiments are conducted for numerous example cases to examine the effectiveness of the developed inverse modeling approach to reconstruct the dynamic traction.

In Chapter 3, the method in the previous chapter, particularly the DTO approach, is further examined for a domain that is truncated by using a WABC. Dynamic traction at a WABC mimics seismic incident wavefield. Numerical experiments are conducted for numerous example cases to test the effectiveness of this method to reconstruct dynamic traction at the WABC.

The methods in Chapter 2 and 3 are limited to a domain, of which material property distribution is known in advance. To examine the feasibility to address this limitation, Chapter 4 investigates the feasibility of identifying both (1) unknown profiles of moving vibrational forces and (2) unknown material profiles of a continuous beam model. This chapter employs the FEM to solve the wave equations of a Timoshenko beam subject to a moving vibrational source. Genetic Algorithm (GA) is used as an inversion solver. The outcomes of the numerical experiments present the feasibility of the joint inversion.

Lastly, in Chapter 5, I summarize the conclusions from the research reported in this dissertation, and discuss future direction. I note that parts of the research presented in this dissertation have already been submitted to journal papers.
CHAPTER 2
FULL-WAVEFORM INVERSION OF INCOHERENT DYNAMIC TRACTION IN A BOUNDED 2D DOMAIN OF SCALAR WAVE MOTIONS.

2.1 Problem Definition
This study is aimed at reconstructing the spatial and temporal distributions of dynamic traction on a boundary of an undamped solid by using measured wave responses at sparsely-distributed sensors on the top surface of the solid (see Fig. 2.1). The geometries and the material properties of the solid are assumed to be known in advance.

2.1.1 The governing equation
The governing equation for the shear (SH) wave propagation in the undamped solid domain is (we omit to show the spatial and/or temporal dependency of the variables):

$$\nabla \cdot (G \nabla u) - \rho \frac{\partial^2 u}{\partial t^2} = 0, \quad \text{on } \Omega \times J,$$

(2.1)

where $u = u(x,y,t)$ denotes the displacement field in the anti-plane ($z$) direction of the wave motion of a solid particle (i.e., SH wave motion); $x$, $y$, and $t$ denote horizontal and vertical coordinates and time; $G(x,y)$ and $\rho(x,y)$ denote the shear modulus and the mass density of the solid; $\Omega$ denotes
Figure 2.1: Problem setting of Chapter 2.
the domain, and $J = (0, T]$ is the time interval of interest. The solid is subject to a traction-free condition on the top surface ($\Gamma_t$) and dynamic shear stress on the bottom surface ($\Gamma_b$):

$$G \frac{\partial u}{\partial y}(x, 0, t) = 0, \quad 0 \leq x \leq L, \quad (2.2)$$

$$G \frac{\partial u}{\partial y}(x, D, t) = F(x, t), \quad 0 \leq x \leq L, \quad (2.3)$$

where $D$ is the $y$-coordinate of $\Gamma_b$, and $F(x, t)$ denotes the dynamic shear stress applied on $\Gamma_b$. The solid is constrained by fixed boundary conditions on the left ($\Gamma_l$) and right ($\Gamma_r$) boundaries:

$$u(0, y, t) = 0, \quad D < y < 0, \quad (2.4)$$

$$u(L, y, t) = 0, \quad D < y < 0. \quad (2.5)$$

where $L$ is the $x$-coordinate of $\Gamma_r$. The governing wave physics is also subject to zero initial-value conditions:

$$u(x, y, 0) = 0, \quad (2.6)$$

$$\frac{\partial u}{\partial t}(x, y, 0) = 0. \quad (2.7)$$

We note that this work considers a 2D bounded domain as a prototype for the seismic-input inversion problem in the multi-dimensional setting. Continuing this work, we will investigate the seismic-input inversion in a 2D/3D unbounded (truncated) domain.

### 2.1.2 Parameterization of an estimated dynamic traction function

We discretize an estimated dynamic traction function, $F(x, t)$, over space and time as:

$$F(x, t) = \sum_{k=1}^{N_x} \sum_{j=1}^{N_t} \Phi_k(x) \phi_j(t) F_{kj}, \quad (2.8)$$
where $\Phi_k(x)$ denotes the $k$-th component of a vector of global basis functions used for the spatial discretization of $F(x,t)$; $\phi_j(t)$ denotes the $j$-th component of a vector of global basis functions used for the temporal discretization of $F(x,t)$; $F_{kj}$ denotes the discretized value of $F(x,t)$ at each discrete location $x_k$ and time $t_j$; and $N_x$ and $N_t$ denote the numbers of discretization points over space and time, respectively. Although the shape functions to construct $\Phi_k(x)$ and $\phi_j(t)$ can be of any low order, linear shape functions are used for both of them in the presented inverse modeling. The sizes of the temporal and spatial discretization are set to be, respectively, the time-step size ($\Delta t$) of the forward time integration and the element size ($\Delta x$) of a mesh on $\Gamma_b$ for a forward wave solver. The presented inverse modeling is aimed at reconstructing the set of control parameters $F_{kj}$, of which corresponding wave responses in the domain are consistent with the measurement on $\Gamma_t$.

### 2.2 Inverse Modeling—the optimize-then-discretize (OTD) approach

This section presents the OTD modeling for identifying the temporal and spatial distributions of unknown traction $F(x,t)$ based on measured wave responses on the top surface of the solid. First, this section presents the mathematical modeling of deriving the first-order optimality conditions in a continuous form. Second, this section shows the discrete forms of the state and adjoint equations and the gradient of the objective functional with respect to the control parameters.

#### 2.2.1 The objective functional

We cast the presented inverse problem into a minimization problem, where we seek the values of control parameters (i.e., $F_{kj}$ in (2.8) for all $k$ and $j$) that correspond to a minimum (either global or local one) of an objective functional:

$$
\mathcal{L} = \int_0^T \sum_{i=1}^{N_x} (u_m - u_i)^2 \, dt + R^{TN},
$$

(2.9)
where $u_m$ denotes the displacement field of the measured wave response at the $i$-th sensor induced by a targeted $F(x,t)$; $u_i$ denotes the computed counterpart due to an estimated $F(x,t)$, which is constructed by estimated control parameters; and $N_s$ denotes the number of sensors. In this computational study, $u_m$ is synthetically created by using a forward wave solver with a pseudo target of $F(x,t)$. The first term of (2.9) is a misfit between $u_m$ and $u_i$. Although the misfit function is a quadratic function in terms of $u$, the relation between $u$ and a set of the control parameters is nonlinear. Thus, the misfit function would be non-convex, and the numerical optimizer may suffer from the solution multiplicity. To address the solution multiplicity, we employed the second term of (2.9), $R^{TN}$, which denotes the Tikhonov (TN) regularization term:

$$R^{TN} = \frac{R}{2} \int_0^T \int_{\Gamma_b} \left( \frac{\partial F(x,t)}{\partial x} \right)^2 + \left( \frac{\partial F(x,t)}{\partial t} \right)^2 d\Gamma dt,$$

(2.10)

where $R$ is the regularization factor, which adjusts the amount of penalty on the derivative of $F(x,t)$. By minimizing the regularization term $R^{TN}$ along with the misfit, we attempt to minimize the discontinuity of $F(x,t)$ and smooth it while mitigating the solution multiplicity of the presented inverse problem. It is well known that, when the material inversion is performed, the TN regularization on a material profile overly smooths the discontinuity at the interface of layered media. Thus, the TN regularization is suited for identifying a smooth material profile while the total variation (TV) is used for enhancing the discontinuity of the material profile. Meanwhile, for the seismic input inversion, a typical time signal of a seismic input motion should be a smooth function because the high-frequency content of a discontinuous signal cannot be retained along the propagation path from a seismic source to a near-surface domain due to attenuation. Thus, this work used the TN regularization under a hypothesis that using the TN regularization can improve the uniqueness of the inversion solution while smoothing the estimated traction function. This hypothesis is tested in Example 4 shown in the later section ‘Numerical Experiments’.
2.2.2 Lagrangian functional

By imposing the governing equation (2.1) and the Neumann boundary condition (2.3) onto the side of the objective functional via Lagrange multipliers, a Lagrangian functional is built as:

\[ A = \sum_{i=1}^{N_s} (u_{mi} - u_i)^2 \, dt + \mathcal{R}^T \mathcal{N} \]

\[ + \int_0^T \int_\Omega \lambda \left[ \nabla \cdot (G \nabla u) - \rho \frac{\partial^2 u}{\partial t^2} \right] \, d\Omega \, dt \]

\[ + \int_0^T \int_{\Gamma_b} \lambda_F \left[ G \frac{\partial u}{\partial y} - F(x, t) \right] \, d\Gamma \, dt, \quad (2.11) \]

where \( \lambda = \lambda(x, y, t) \) and \( \lambda_F = \lambda_F(x, t) \) are the Lagrange multipliers. Note that the boundary condition and initial conditions are implicitly imposed in (2.11): they are not shown in (2.11) but used for the derivation of the adjoint and control equations. The first-order optimality conditions of the Lagrangian functional lead to state, adjoint, and control equations. The satisfaction of these equations leads to an optimal solution, corresponding to the minimal value of the objective functional.

2.2.3 The first-order optimality conditions

The first-order optimality conditions of the Lagrangian functional \( \mathcal{A} \) require the vanishing variations of \( \mathcal{A} \) with respect to the state variable \( u(x, y, t) \), the Lagrange variables \( \lambda(x, y, t) \) and \( \lambda_F(x, t) \), and the control parameter \( \xi = F_{kj} \). Such vanishing conditions lead to a triad of state, adjoint, and control equations [48]:

\[ \delta_{\lambda, \lambda_F} \mathcal{A} = 0 : \quad \text{The first condition (state problem),} \quad (2.12) \]

\[ \delta_{u} \mathcal{A} = 0 : \quad \text{The second condition (adjoint problem),} \quad (2.13) \]

\[ \delta_{\xi} \mathcal{A} = 0 : \quad \text{The third condition (control problem).} \quad (2.14) \]
2.2.3.1 The first condition

For the first condition, the variation of $A$ with respect to the Lagrange variables $\lambda(x,y,t)$ and $\lambda_F(x,t)$ vanishes when the state problem—the original governing wave equation (2.1) and its associated boundary and initial-value conditions—is satisfied. Our inverse modeling procedure automatically satisfies it by numerically solving the state problem for estimated control parameters.

2.2.3.2 The second condition

As the second condition, the variation of $A$ with respect to the state variable $u(x,y,t)$ should vanish as:

$$
\delta_u A = \int_0^T \delta_u \sum_{i=1}^{N_s} (u_{m_i} - u_i)^2 \, dt + \int_0^T \int_{\Omega} \lambda \nabla \cdot (G \nabla \delta u) \, d\Omega \, dt - \int_0^T \int_{\Omega} \lambda \rho \frac{\partial^2 \delta u}{\partial t^2} \, d\Omega \, dt \\
+ \int_0^T \int_{\Gamma_b} \lambda_F \frac{\partial \delta u}{\partial y} \, d\Gamma \, dt - \int_0^T \int_{\Gamma_b} \lambda_F F(x,t) \, d\Gamma \, dt = 0. \tag{2.15}
$$

Part $a$ in (2.15) can be written as:

$$
a = \int_0^T \delta_u \sum_{i=1}^{N_s} (u_{m_i} - u_i)^2 \, dt = -\int_0^T \sum_{i=1}^{N_s} 2(u_{m_i} - u_i) \delta u_i \, dt \\
= -\int_0^T \int_{\Omega} 2(u_m - u) \sum_{i=1}^{N_s} \Delta(x-x_i,y-y_i) \, d\Omega \, dt, \tag{2.16}
$$

where $\Delta(x-x_i,y-y_i)$ is the Dirac delta function. Integrating part $b$ in (2.15) by parts over space twice leads to:

$$
b = \int_0^T \int_{\Omega} \lambda \nabla \cdot (G \nabla \delta u) \, d\Omega \, dt \\
= \int_0^T \int_{\Omega} \nabla \cdot (\lambda G \nabla \delta u) \, d\Omega \, dt - \int_0^T \int_{\Omega} (\nabla \lambda \cdot G \nabla \delta u) \, d\Omega \, dt \\
= \int_0^T \int_{\Omega} \nabla \cdot (\lambda G \nabla \delta u) \, d\Omega \, dt + \int_0^T \int_{\Omega} \delta u \nabla \cdot (G \nabla \lambda) \, d\Omega \, dt - \int_0^T \int_{\Omega} \nabla \cdot (\delta u G \nabla \lambda) \, d\Omega \, dt. \tag{2.17}
$$
Due to the Divergence Theorem, (2.17) becomes:

\[ b = \int_0^T \int \lambda G \frac{\partial \delta u}{\partial n} \, d\Gamma \, dt + \int_0^T \int \delta u \nabla \cdot (G \nabla \lambda) \, d\Omega \, dt - \int_0^T \int \delta u G \frac{\partial \lambda}{\partial n} \, d\Gamma \, dt \]

\[ = \int_0^T \left[ \int_{\Gamma_t} \lambda G \frac{\partial \delta u}{\partial n} \, d\Gamma + \int_{\Gamma_b} \lambda G \frac{\partial \delta u}{\partial n} \, d\Gamma + \int_{\Gamma_{lr}} \lambda G \frac{\partial \delta u}{\partial n} \, d\Gamma \right] \, dt \]

\[ + \int_0^T \int \delta u \nabla \cdot (G \nabla \lambda) \, d\Omega \, dt - \int_0^T \left[ \int_{\Gamma_{tb}} \delta u G \frac{\partial \lambda}{\partial n} \, d\Gamma + \int_{\Gamma_{lr}} \delta u G \frac{\partial \lambda}{\partial n} \, d\Gamma \right] \, dt \]

\[ = - \int_0^T \int_{\Gamma_b} \lambda G \frac{\partial \delta u}{\partial y} \, d\Gamma \, dt + \int_0^T \int_{\Gamma_{lr}} \lambda G \frac{\partial \delta u}{\partial n} \, d\Gamma \, dt \]

\[ + \int_0^T \int \delta u \nabla \cdot (G \nabla \lambda) \, d\Omega \, dt - \int_0^T \int_{\Gamma_{tb}} \delta u G \frac{\partial \lambda}{\partial n} \, d\Gamma \, dt. \tag{2.18} \]

where \( \frac{\partial (\cdot)}{\partial n} \) denotes a directional derivative of a variable in the direction of an outward unit normal vector \( n \) on \( \Gamma \). Integrating \( c \) in (2.15) by parts over time twice leads to:

\[ c = -\int_0^T \int_{\Omega} \rho \lambda \frac{\partial^2 \delta u}{\partial t^2} \, d\Omega \, dt \]

\[ = -\int_{\Omega} \rho \left[ \lambda \frac{\partial \delta u}{\partial t} \right]_0^T \, d\Omega + \int_{\Omega} \int \rho \frac{\partial \lambda}{\partial t} \frac{\partial \delta u}{\partial t} \, d\Omega \, dt \]

\[ = -\int_{\Omega} \rho \left[ \lambda \frac{\partial \delta u}{\partial t} \right]_0^T \, d\Omega + \int_{\Omega} \rho \frac{\partial \lambda}{\partial t} \delta u \, d\Omega - \int_0^T \int_{\Omega} \rho \frac{\partial^2 \lambda}{\partial t^2} \delta u \, d\Omega \, dt, \tag{2.19} \]

Because \( \frac{\partial \delta u}{\partial t} (t = 0) \) and \( \delta u(t = 0) \) vanish, (2.19) becomes:

\[ c = -\int_{\Omega} \left[ \rho \lambda \frac{\partial \delta u}{\partial t} \right]_T \, d\Omega + \int_{\Omega} \left[ \rho \frac{\partial \lambda}{\partial t} \delta u \right]_T \, d\Omega - \int_0^T \int_{\Omega} \rho \frac{\partial^2 \lambda}{\partial t^2} \delta u \, d\Omega \, dt. \tag{2.20} \]

Due to (2.16), (2.18), and (2.20), (2.15) can be written as:
\[
\delta_u \mathcal{A} = \int_0^T \int_{\Gamma_b} (\lambda_F - \lambda) G \frac{\partial \delta u}{\partial y} \, d\Gamma \, dt \\
+ \int_0^T \int_{\Gamma_{lb}} \lambda G \frac{\partial \delta u}{\partial n} \, d\Gamma \, dt \\
- \int_0^T \int_{\Gamma_{tb}} \delta u G \frac{\partial \lambda}{\partial n} \, d\Gamma \, dt \\
- \int_0^T \int_\Omega \left[ \rho \lambda \frac{\partial \delta u}{\partial t} \right]_T \, d\Omega \\\n+ \int_0^T \int_\Omega \delta u \left[ -2(u_m - u) \sum_{i=1}^{N_s} \Delta(x - x_i, y - y_i) + \nabla \cdot (G \nabla \lambda) - \rho \frac{\partial^2 \lambda}{\partial t^2} \right] \, d\Omega \, dt = 0. \quad (2.21)
\]

Such a vanishing variational condition leads to the following adjoint equation:

\[
\nabla \cdot (G \nabla \lambda) - \rho \frac{\partial^2 \lambda}{\partial t^2} = 2(u_m - u) \sum_{i=1}^{N_s} \Delta(x - x_i, y - y_i), \quad \text{on } \Omega \times [0, T), \quad (2.22)
\]

where \( \Delta(x - x_i, y - y_i) \) is the Dirac delta. In the adjoint PDE, the difference between \( u_m \) and \( u \) serves as the time signal of a point wave source at each location of a sensor. It is noteworthy that the strong form of the adjoint PDE is derived from the weak-form like equation (2.21). Although the strong form of the adjoint PDE is weakly satisfied, the FEM solution of the adjoint PDE fully satisfies the weak-form like equation (2.21) and, thus, satisfies the second condition of the first-order optimality condition. On the other hand, if the Lagrangian functional is built by imposing the discrete form of the state PDE, the aforementioned issue does not arise because the corresponding discrete form of the adjoint problem fully satisfies the second condition of the first-order optimality condition. To study the latter aspect, this paper also presents the discretize-then-optimize (DTO) counterpart in the later section ‘Inverse Modeling—the discretize–then-optimize (DTO) approach’, and our numerical experiments tests its inversion performance in Example 2 in the later section ‘Numerical Experiments’.

The adjoint PDE is also subject to the following boundary conditions:

\[
\begin{align*}
\lambda(0,y,t) &= \lambda(L,y,t) = 0, \quad D < y < 0, \\
\frac{\partial \lambda}{\partial y}(x,0,t) &= \frac{\partial \lambda}{\partial y}(x,D,t) = 0, \quad 0 \leq x \leq L,
\end{align*}
\quad (2.23)
\]
and the following final-value conditions:

\[
\lambda(x, y, T) = 0, \\
\frac{\partial \lambda}{\partial t}(x, y, T) = 0.
\]  

(2.24)

Finally, \( \delta_u \mathcal{A} = 0 \) in (2.21) is satisfied when we satisfy the adjoint problem in (2.22) to (2.24).

2.2.3.3 The third condition

The third condition states that the variation of \( \mathcal{A} \) with respect to a scalar-valued control parameter \( \xi = F_{kj} \) should vanish as:

\[
\delta_\xi \mathcal{A} = \frac{\partial \mathcal{A}}{\partial \xi} = - \int_0^T \sum_{i=1}^{N_s} \left[ 2(u_{mi} - u_i) \left( \frac{\partial u_i}{\partial \xi} \right) \right] dt + \int_0^T \int_\Omega \left( \nabla \cdot (G \nabla u) - \rho \frac{\partial^2 u}{\partial t^2} \right) d\Omega dt - \int_0^T \int_{\Gamma_b} \lambda F \frac{d}{dy} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt
\]

\[
= - \int_0^T \int_\Omega \rho \lambda \frac{\partial^2 u}{\partial t^2} \left( \frac{\partial u}{\partial \xi} \right) d\Omega dt + \int_0^T \int_{\Gamma_b} \lambda F \frac{d}{dy} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt - \int_0^T \int_{\Gamma_b} \lambda F \frac{d}{d\xi} (F(x, t)) d\Gamma dt
\]

\[
+ \partial \mathcal{A}^{TN} \frac{\partial \mathcal{A}}{\partial \xi} = 0.
\]  

(2.25)

Part \( e \) in (2.25) can be written as:

\[
e = - \int_0^T \int_\Omega 2(u_m - u) \left( \frac{\partial u}{\partial \xi} \right) \sum_{i=1}^{N_s} \Delta(x - x_i, y - y_i) d\Omega dt.
\]  

(2.26)
Integrating $f$ in (2.25) by parts twice over space leads to:

$$
f = -\int_0^T \int_{\Gamma_b} \lambda G \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt + \int_0^T \int_{\Gamma_l} \lambda G \frac{\partial u}{\partial n} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt + \int_0^T \int_{\Omega} \frac{\partial u}{\partial \xi} \nabla \cdot (G \nabla \lambda) d\Omega - \int_0^T \int_{\Gamma_{lb}} \frac{\partial^2 u}{\partial n \partial \xi} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt. \tag{2.27}
$$

By integrating $g$ in (2.25) by parts over time twice and knowing that $\frac{\partial}{\partial t} \frac{\partial u}{\partial \xi}(t = 0)$ and $\frac{\partial u}{\partial \xi}(t = 0)$ vanish, we rewrite $g$ in (2.25) as:

$$
g = -\int_\Omega \left[ \rho \lambda \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial \xi} \right) \right] T d\Omega + \int_\Omega \left[ \rho \frac{\partial \lambda}{\partial t} \left( \frac{\partial u}{\partial \xi} \right) \right] T d\Omega - \int_0^T \int_\Omega \rho \frac{\partial^2 \lambda}{\partial t^2} \left( \frac{\partial u}{\partial \xi} \right) d\Omega dt. \tag{2.28}
$$

Due to (2.26), (2.27), and (2.28), (2.25) becomes:

$$
\delta \xi = \int_0^T \int_{\Gamma_b} (\lambda_F - \lambda) G \frac{\partial u}{\partial y} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt - \int_0^T \int_{\Gamma_l} \lambda G \frac{\partial u}{\partial n} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt + \int_0^T \int_{\Gamma_{lb}} \lambda G \frac{\partial \lambda}{\partial n} \left( \frac{\partial u}{\partial \xi} \right) d\Gamma dt
\nonumber
- \int_\Omega \left[ \rho \lambda \frac{\partial}{\partial t} \left( \frac{\partial u}{\partial \xi} \right) \right] T d\Omega + \int_\Omega \left[ \rho \frac{\partial \lambda}{\partial t} \left( \frac{\partial u}{\partial \xi} \right) \right] T d\Omega
\nonumber
+ \int_0^T \int_\Omega \frac{\partial u}{\partial \xi} \left[ -2(u_m - u) \sum_{i=1}^{N_s} \Delta(x - x_i, y - y_i) + \nabla \cdot (G \nabla \lambda) - \rho \frac{\partial^2 \lambda}{\partial t^2} \right] d\Omega dt
\nonumber
- \int_0^T \int_{\Gamma_b} \lambda_F \frac{\partial}{\partial \xi} (F(x, t)) d\Gamma dt + \frac{\partial \mathcal{R}^{TN}}{\partial \xi} = 0. \tag{2.29}
$$

Eq. (2.29) reduces to:

$$
\delta \xi = -\int_0^T \int_{\Gamma_b} \lambda \frac{\partial}{\partial \xi} (F(x, t)) d\Gamma dt + \frac{\partial \mathcal{R}^{TN}}{\partial \xi} = 0. \tag{2.30}
$$
The first term of (2.30) is:

\[ - \int_0^T \int_{\Gamma_b} \lambda \frac{\partial}{\partial \xi} (F(x,t)) \, d\Gamma \, dt = - \int_0^T \int_{\Gamma_b} \lambda \Phi_k(x) \phi_j(t) \, d\Gamma \, dt \]  

(2.31)

In addition, the second term in (2.30) is:

\[
\frac{\partial \mathcal{R}^{TN}}{\partial \xi} = \frac{\partial}{\partial \xi} \left( \frac{R}{2} \int_0^T \int_{\Gamma_b} \left( \frac{\partial F(x,t)}{\partial x} \right)^2 + \left( \frac{\partial F(x,t)}{\partial t} \right)^2 \, d\Gamma \, dt \right) \\
= R \int_0^T \int_{\Gamma_b} \left( \frac{\partial F \partial \hat{F}}{\partial x \partial x} + \left( \frac{\partial F \partial \hat{F}}{\partial t \partial t} \right) d\Gamma \, dt \right) \\
= \left[ R \int_0^T \int_{\Gamma_b} \frac{\partial F \partial \hat{F}}{\partial x} \, d\Gamma \, dt \right]_0^T - R \int_0^T \int_{\Gamma_b} \frac{\partial^2 F \partial \hat{F}}{\partial x^2} \, d\Gamma \, dt \\
+ \left[ R \int_{\Gamma_b} \frac{\partial F \partial \hat{F}}{\partial t} \, d\Gamma \right]_0^T - R \int_0^T \int_{\Gamma_b} \frac{\partial^2 F \partial \hat{F}}{\partial t^2} \, d\Gamma \, dt \\
= - R \int_0^T \int_{\Gamma_b} \left( \frac{\partial^2 F(x,t)}{\partial x^2} + \frac{\partial^2 F(x,t)}{\partial t^2} \right) \Phi_k(x) \phi_j(t) \, d\Gamma \, dt
\]  

(2.32)

where \( \hat{F} \) is defined as:

\[ \hat{F} = \frac{\partial F(x,t)}{\partial \xi} = \Phi_k(x) \phi_j(t), \]  

(2.33)

and we enforce that:

\[
\frac{\partial F}{\partial x} = 0, \quad \text{at} \; x = 0, L, \\
\frac{\partial F}{\partial t} = 0, \quad \text{at} \; t = 0, T.
\]  

(2.34)

Therefore, the vanishing variation condition leads to the following control equation:
\[ \delta_{\xi} \mathcal{A} = \frac{\partial \mathcal{A}}{\partial \xi} \]

\[ = - \int_{0}^{T} \int_{\Gamma_{b}} \lambda \Phi_{k}(x) \phi_{j}(t) d\Gamma_{b} dt - R \int_{0}^{T} \int_{\Gamma_{b}} \left( \frac{\partial^{2} F(x,t)}{\partial x^{2}} + \frac{\partial^{2} F(x,t)}{\partial t^{2}} \right) \Phi_{k}(x) \phi_{j}(t) d\Gamma_{b} dt = 0. \]

(2.35)

Note that \( \delta_{\xi} \mathcal{A} = \frac{\partial \mathcal{A}}{\partial \xi} \) is the derivative of \( \mathcal{A} \) with respect to a control parameter \( \xi = F_{kj} \). Since the side-imposed terms in \( \mathcal{A} \) vanish, \( \frac{\partial \mathcal{A}}{\partial \xi} \) is equivalent to \( \frac{\partial \mathcal{L}}{\partial \xi} \), which constitutes a gradient vector \( \nabla_{\xi} \mathcal{L} \), where \( \xi \) is a vector of all the control parameters. The control equation (2.35) implies that \( \nabla_{\xi} \mathcal{L} \) at any estimated values of \( \xi \) can be evaluated in a semi-analytical manner by using its closed form once the solutions of state and adjoint problems are computed.

### 2.2.4 Finite element solution of the state problem

To find \( u \in H^{1}(\Omega) \times J \), we cast the weak form of the state problem as:

\[ \int_{\Omega} \nabla v \cdot (G \nabla u) d\Omega + \int_{\Omega} \rho \frac{\partial^{2} u}{\partial t^{2}} d\Omega = - \int_{\Gamma_{b}} v F(x,t) d\Gamma_{b}, \]

(2.36)

where \( v(x,y) \in H^{1}(\Omega) \) denotes a test function. The function space for a scalar-valued \( u \) (or \( v \)) is defined as:

\[ H^{1}(\Omega) = \left\{ u : \int_{\Omega} |\nabla u|^{2} d\Omega < \infty, \quad u|_{\Gamma_{r}} = 0 \right\}. \]

(2.37)

To resolve the weak form numerically, we use the standard finite-element approximation. We approximate the test and trial functions, respectively, as:

\[ v(x,y) \simeq v^{T} \psi(x,y), \quad u(x,y,t) \simeq \psi(x,y)^{T} u(t), \]

(2.38)

where \( \psi(x,y) \) denotes a vector of global basis functions constructed by shape functions of each finite element mesh in the domain, and \( u(t) \) denotes a vector of nodal solutions of the state problem.
Then, (2.36) reduces to the following discrete form:

\[ \mathbf{M} \ddot{\mathbf{u}}(t) + \mathbf{Ku}(t) = \mathbf{F}(t), \]  

(2.39)

where \( \ddot{\cdot} \) denotes the second-order derivative of its subtended variable with respect to \( t \); \( \mathbf{M} \) denotes a global mass matrix; \( \mathbf{K} \) denotes a global stiffness matrix; \( \mathbf{F}(t) \) denotes a global force vector. They are defined as:

\[
\mathbf{K} = \int_{\Omega} G \left( \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial y} \right) d\Omega,
\]

\[
\mathbf{M} = \int_{\Omega} \rho \psi \psi^T d\Omega,
\]

\[
\mathbf{F}(t) = -\int_{\Gamma_b} \psi \mathbf{F}(x,t) d\Gamma.
\]

(2.40)

2.2.5 Finite element solution of the adjoint problem

To find \( \lambda \in H^1(\Omega) \times [0,T) \), the weak form of the adjoint equation (2.22) is obtained as:

\[
\int_{\Omega} \nabla v \cdot (G \nabla \lambda) d\Omega + \int_{\Omega} \nu \rho \frac{\partial^2 \lambda}{\partial t^2} d\Omega = -\sum_{i=1}^{N_A} 2v(x_i,y_i)(u_m(x_i,y_i,t) - u(x_i,y_i,t)).
\]

(2.41)

The test and trial functions are approximated as follows:

\[
v(x,y) \simeq v^T \psi(x,y), \quad \lambda(x,y,t) \simeq \psi(x,y)^T \lambda(t),
\]

(2.42)

where \( \lambda(t) \) is a vector of the nodal adjoint solution. The weak form of the adjoint problem changes to the following time-dependent discrete form:

\[ \mathbf{M} \ddot{\lambda}(t) + \mathbf{K}\lambda(t) = \mathbf{F}_{\text{adj}}(t), \]

(2.43)
where $F_{\text{adj}}(t)$ is defined as:

$$F_{\text{adj}}(t) = 2 \sum_{i=1}^{N_s} (\psi(x_i, y_i)u(x_i, y_i, t) - \psi(x_i, y_i)u_m(x_i, y_i, t)).$$  

(2.44)

We note that the specific forms of the matrices $K$ and $M$ in the discrete form of the adjoint problem in (2.43) are identical to those for the state problem in (2.40).

### 2.2.6 Time integration

We solve the time-dependent discrete form of the state problem in (2.39) by using the implicit Newmark time integration (i.e., average-acceleration scheme). We omit to show the detail of the forward time integration procedure of the state problem.

On the other hand, for the time-dependent discrete form of the adjoint problem in (2.43), the time integration begins from the final time $t = T$ and ends at the initial time $t = 0$. That is, the backward time integration begins with the final-value conditions, $\lambda(T) = 0$, $\dot{\lambda}(T) = 0$, and

$$\ddot{\lambda}(T) = M^{-1}F_{\text{adj}}(T).$$  

(2.45)

For the backward time-marching procedure from $t = T$, the following approximations are used:

$$\dot{\lambda}_n = \dot{\lambda}_{n+1} - \frac{\Delta t}{2} \ddot{\lambda}_n - \frac{\Delta t}{2} \ddot{\lambda}_{n+1},$$  

(2.46)

$$\ddot{\lambda}_n = \frac{4}{(\Delta t)^2} (\lambda_{n+1} - \lambda_n) - \frac{4}{\Delta t} \dot{\lambda}_n - \ddot{\lambda}_{n+1}.$$  

(2.47)

Substituting (2.46) into (2.47) yields:

$$\ddot{\lambda}_n = -\frac{4}{(\Delta t)^2} (\lambda_{n+1} - \lambda_n) + \frac{4}{\Delta t} \dot{\lambda}_{n+1} - \ddot{\lambda}_{n+1}.$$  

(2.48)
By inserting (2.48) into the discrete form of the adjoint equation (2.43), we obtain the following equation that will be used for computing $\lambda_n$ at every $n$-th time step:

$$
\lambda_n = \left[ K + \frac{4}{(\Delta t)^2} M \right]^{-1} \left\{ F_{\text{adj}}(t_n) + M \left( \frac{4}{(\Delta t)^2} \lambda_{n+1} - \frac{4}{\Delta t} \dot{\lambda}_{n+1} + \ddot{\lambda}_{n+1} \right) \right\}.
$$

(2.49)

Once we obtain $\lambda_n$ from (2.49), we obtain $\ddot{\lambda}_n$ from (2.48) and, in turn, $\dot{\lambda}_n$ from (2.46).

### 2.2.7 The discrete form of the gradient

The gradient of the objective functional $\mathcal{L}$ with respect to a scalar variable $\xi$ can now be numerically computed as:

$$
\nabla_{(\xi=F_{kj})} \mathcal{L} = -\Delta x \Delta t \times \lambda(x_k, D, t_j) - R \times \Delta x \Delta t \times \left[ \frac{\partial^2 F(x, t)}{\partial x^2} + \frac{\partial^2 F(x, t)}{\partial t^2} \right] \text{ at } x_k, t_j.
$$

(2.50)

Here, this work uses the aforementioned FEM solution of the adjoint problem for evaluating the gradient in (2.50) under the OTD approach.

### 2.3 Inverse Modeling—the discretize–then-optimize (DTO) approach

This section presents the inverse modeling based on the DTO approach. The Lagrangian functional is built by imposing the discrete form of the state problem, using the discrete adjoint variable, into the objective functional in the discrete form. The first-order optimality conditions are derived in the discrete form. The time-integration implementation of the state and adjoint problems are already embedded in their discrete forms.

#### 2.3.1 The discrete objective functional

The discrete-form counterpart of the objective functional (2.9) is given by:

$$
\mathcal{L} = (\hat{u}_m - \hat{u})^T \bar{B} (\hat{u}_m - \hat{u}) + \frac{R}{2} \hat{F}^T R \hat{F},
$$

(2.51)
where \( \hat{u} = [u_0 \hat{u}_0 u_1 \hat{u}_1 \ldots u_N \hat{u}_N]^T \) corresponds to the space-time discretization of \( u(x,y,t) \) for \( (x,y) \in \Omega \) and \( t \in [0,T] \), induced by an estimated \( F(x,t) \) (\( N \) is the number of time steps, and \( u_i \) are the spatial degrees of freedom at the \( i \)-th time step); \( \hat{u}_m \) is the space-time discretization of \( u_m(x,t) \) induced by a targeted \( F(x,t) \); and \( B \) is a block diagonal matrix, determined as \( B = \Delta t B \) on the diagonal, where \( B \) is a square matrix that is zero everywhere except on the diagonals that correspond to a degree of freedom for which measured data are available; \( R \) is the regularization factor; \( \hat{F} = [00 \hat{F}_00 \ldots \hat{F}_N00]^T \) is a global force vector corresponding to all the time steps—that is, the discrete control parameter \( F_{kj} \) are populated in \( \hat{F} \); and \( R \) is the matrix corresponding to the discretization scheme used for the regularization terms defined as:

\[
R = \int_0^T \int_{\Gamma_b} \left( \frac{\partial w(x,0,t)}{\partial x} \frac{\partial w^T(x,0,t)}{\partial x} + \frac{\partial w(x,0,t)}{\partial t} \frac{\partial w^T(x,0,t)}{\partial t} \right) d\Gamma dt, \tag{2.52}
\]

where \( w(x,y,t) \) denotes a vector of global basis functions, in both space and time, constructed by shape functions of each finite element mesh in the domain and the shape functions over the time. That is, an estimated traction function \( F(x,t) \) can be discretized as:

\[
F(x,t) = w^T(x,y = 0,t)\hat{F}. \tag{2.53}
\]

### 2.3.2 The discrete Lagrangian functional

The Lagrangian functional corresponding to (2.51) is built by imposing the discrete form of the state problem using the discrete adjoint variable:

\[
\mathcal{L} = (\hat{u}_m - \hat{u})^T B (\hat{u}_m - \hat{u}) + \frac{R^T}{2} \hat{F} \hat{F}^T + \hat{\lambda}^T (Q \hat{u} - \hat{F}), \tag{2.54}
\]

where \( \hat{\lambda} = [\lambda_0 \hat{\lambda}_0 \lambda_1 \hat{\lambda}_1 \ldots \lambda_N \hat{\lambda}_N \hat{\lambda}_N]^T \) is the discrete (space-time) Lagrange multiplier that enforces the discrete forward problem as a constraint; and \( Q \) is the discrete forward operator defined as:
\[ Q = \begin{bmatrix}
I & 0 & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & I & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
K & 0 & M & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
L_1 & L_2 & L_3 & \text{Keff} & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & 0 & 0 & 0 & \ldots & L_1 & L_2 & L_3 & \text{Keff} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & a_1 I & I & 0 & -a_1 I & I & 0 \\
0 & 0 & 0 & 0 & 0 & \ldots & a_0 I & a_2 I & I & -a_0 I & 0 & I \\
\end{bmatrix}, \quad (2.55) \]

where:

\[
a_0 = \frac{4}{(\Delta t)^2}, \quad a_1 = \frac{2}{\Delta t}, \quad a_2 = \frac{4}{\Delta t}, \quad (2.56)
\]

\[
\text{Keff} = a_0 M + K, \quad (2.57)
\]

\[
L_1 = -a_0 M, \quad L_2 = -a_2 M, \quad L_3 = -M. \quad (2.58)
\]

### 2.3.3 The first-order optimality condition in the DTO modeling

The discrete optimality conditions of (2.51) require that the variations of \( \mathcal{J} \) with respect to \( \hat{\lambda} \), \( \hat{u} \) and \( \hat{F} \) vanish. The first condition, taking the variation with respect to \( \hat{\lambda} \), recovers the discrete form of the state equation:

\[
\frac{\partial \mathcal{J}}{\partial \hat{\lambda}} = Q \hat{u} - \hat{F} = 0. \quad (2.59)
\]
For the second condition, the variation of $\mathcal{A}$ with respect to $\hat{\mathbf{u}}$ should vanish:

$$\frac{\partial \mathcal{A}}{\partial \hat{\mathbf{u}}} = \mathbf{Q}^T \hat{\mathbf{\lambda}} + 2 \mathbf{B} (\hat{\mathbf{u}}_m - \hat{\mathbf{u}}) = 0. \tag{2.60}$$

Equation (2.60) represents the discrete adjoint equation. Since it involves the transpose of $\mathbf{Q}$, we solve it by marching backwards in time. For example, from the last two rows of (2.55), we obtain the final conditions:

$$\ddot{\mathbf{\lambda}}_N = 0, \tag{2.61}$$
$$\dot{\mathbf{\lambda}}_N = 0, \tag{2.62}$$

respectively; and the third row from the bottom yields:

$$\mathbf{K}_{\text{eff}}^T \mathbf{\lambda}_N = 2\Delta t \mathbf{B} (\mathbf{u}_N - \mathbf{u}_{mN}) + a_1 \dot{\mathbf{\lambda}}_N + a_0 \ddot{\mathbf{\lambda}}_N, \tag{2.63}$$

which can be solved for $\mathbf{\lambda}_N$. For time steps $i = N - 1, N - 2, \ldots, 1$, we first update $\ddot{\mathbf{\lambda}}_i$ and $\dot{\mathbf{\lambda}}_i$ as the following:

$$\ddot{\mathbf{\lambda}}_i = \mathbf{M}^T \mathbf{\lambda}_{i+1} - \ddot{\mathbf{\lambda}}_{i+1}, \tag{2.64}$$
$$\dot{\mathbf{\lambda}}_i = (a_2 \mathbf{M}^T + \mathbf{C}^T) \mathbf{\lambda}_{i+1} - \dot{\mathbf{\lambda}}_{i+1} - a_2 \ddot{\mathbf{\lambda}}_{i+1}, \tag{2.65}$$

and, then, solve the following:

$$\mathbf{K}_{\text{eff}}^T \mathbf{\lambda}_i = 2\Delta t \mathbf{B} (\mathbf{u}_i - \mathbf{u}_{mi}) + a_1 \dot{\mathbf{\lambda}}_i + a_0 \ddot{\mathbf{\lambda}}_i + (a_0 \mathbf{M}^T + a_1 \mathbf{C}^T) \mathbf{\lambda}_{i+1} - a_1 \dot{\mathbf{\lambda}}_{i+1} - a_0 \ddot{\mathbf{\lambda}}_{i+1}, \tag{2.66}$$
Finally, the first three rows of (2.22) result in the following equations. First, we solve:

\[ M^T \ddot{\lambda}_0 = M^T \lambda_1 - \dot{\lambda}_1, \quad (2.67) \]

and, then, update \( \dot{\lambda}_0 \) and \( \lambda_0 \) as the following:

\[
\begin{align*}
\dot{\lambda}_0 &= -C^T \lambda_0 + (a_2 M^T + C^T) \lambda_1 - \dot{\lambda}_1 - a_2 \dot{\lambda}_1, \\
\lambda_0 &= -K^T \lambda_0 + (a_0 M^T + a_1 C^T) \lambda_1 - a_1 \dot{\lambda}_1 - a_0 \dot{\lambda}_1 + \Delta t B (u_0 - u_{m0}).
\end{align*}
\quad (2.68, 2.69) \]

We note that the backward adjoint time integration in the DTO approach differs from that shown in its counterpart of the OTD approach.

The third condition states that the variation of \( \mathcal{A} \) with respect to \( \hat{F} \) should vanish:

\[
\frac{\partial \mathcal{A}}{\partial \hat{F}} = R R \hat{F} - \dot{\lambda} = 0, \quad (2.70)
\]

which represents the discrete control equation and implies that \( \frac{\partial \mathcal{A}}{\partial \xi} = \frac{\partial \hat{\mathcal{A}}}{\partial \xi} \) is the component of the vector:

\[
R R \hat{F} - \dot{\lambda},
\quad (2.71)
\]

at its row corresponding to \( \xi = F_{kj} \).

### 2.3.4 Implementation of the regularization term in the gradient

This subsection shows the detail of evaluating the regularization term, \( R R \hat{F} \), in (2.71). The non-zero contribution of an element (shown in Fig. 2.2) in the space in terms of \( x \) and \( t \) to \( R \) matrix is:
Accordingly, the four elements surrounding $F_{kj}$ (see Fig. 2.2) contribute to the $9 \times 9$ submatrix (i.e., $R^{4E}$) of $R$.

Figure 2.2: $F_{kj}$ is surrounded by four elements in the space in terms of $x$ and $t$: the horizontal and vertical axes represent, respectively, the $x$ coordinate and time $t$.

Then, the component of $R\mathbf{R}^F$ corresponding to $F_{kj}$ can be computed as:

$$\frac{\partial \mathbf{R}^{TN}}{\partial F_{kj}} = R(R_{5-th\ row}^{4E})F^{4E},$$

where the 5-th row of $R^{4E}$ (i.e., $R_{5-th\ row}^{4E}$) is:
\[ R_{5-\text{th row}} = \Delta x \Delta t \]

\[
\begin{bmatrix}
  \left(- \frac{1}{6(\Delta x)^2} - \frac{1}{6(\Delta t)^2}\right) \\
  \left(\frac{1}{3(\Delta x)^2} - \frac{2}{3(\Delta t)^2}\right) \\
  \left(- \frac{1}{6(\Delta x)^2} - \frac{1}{6(\Delta t)^2}\right) \\
  \left(- \frac{2}{3(\Delta x)^2} + \frac{1}{3(\Delta t)^2}\right) \\
  \left(\frac{4}{3(\Delta x)^2} + \frac{4}{3(\Delta t)^2}\right) \\
  \left(- \frac{2}{3(\Delta x)^2} + \frac{1}{3(\Delta t)^2}\right) \\
  \left(- \frac{1}{6(\Delta x)^2} - \frac{1}{6(\Delta t)^2}\right) \\
  \left(\frac{1}{3(\Delta x)^2} - \frac{2}{3(\Delta t)^2}\right) \\
  \left(- \frac{1}{6(\Delta x)^2} - \frac{1}{6(\Delta t)^2}\right)
\end{bmatrix}^T,
\]

and \( F^{4E} \) is defined as:

\[
F^{4E} =
\begin{bmatrix}
  F_{(k-1)(j-1)} \\
  F_{(k-1)j} \\
  F_{(k-1)(j+1)} \\
  F_{k(j-1)} \\
  F_{kj} \\
  F_{k(j+1)} \\
  F_{(k+1)(j-1)} \\
  F_{(k+1)j} \\
  F_{(k+1)(j+1)}
\end{bmatrix}.
\]

Thus, under the DTO approach, (2.73) can be implemented as:
\[
\frac{\partial R^{TN}}{\partial F_{kj}}_{DTO} = - R \times \Delta x \Delta t \times \left\{ \frac{1}{6} \left( \frac{F_{(k-1)(j-1)} - 2F_{(k-1)j} + F_{(k-1)(j+1)}}{\Delta x^2} \right) + \frac{4}{6} \left( \frac{F_{(k-1)j} - 2F_{kj} + F_{(k+1)j}}{\Delta x^2} \right) + \frac{1}{6} \left( \frac{F_{(k-1)(j+1)} - 2F_{k(j+1)} + F_{(k+1)(j+1)}}{\Delta x^2} \right) + \frac{4}{6} \left( \frac{F_{(k-1)j} - 2F_{kj} + F_{(k+1)j}}{\Delta t^2} \right) + \frac{1}{6} \left( \frac{F_{(k-1)(j+1)} - 2F_{k(j+1)} + F_{(k+1)(j+1)}}{\Delta t^2} \right) \right\},
\]

(2.76)

while, its counterpart in the OTD approach can be implemented as:

\[
\frac{\partial R^{TN}}{\partial F_{kj}}_{OTD} = - R \times \Delta x \Delta t \times \left\{ \frac{F_{k(j-1)} - 2F_{kj} + F_{k(j+1)}}{\Delta x^2} + \frac{F_{(k-1)j} - 2F_{kj} + F_{(k+1)j}}{\Delta t^2} \right\},
\]

(2.77)

which corresponds to (2.50).

### 2.4 Numerical Implementation of the Inversion Process

By utilizing the semi-analytically evaluated gradient vector \( \nabla_{\xi} L \), this work iteratively updates a set of estimated control parameters by using the gradient-based minimization scheme as follows:

(a) First, we compute synthetic measured data \( u_m \) at sensors by using pseudo-target traction \( F(x,t) \).

(b) Then, \( u(x,t) \) is obtained by using estimated \( F(x,t) \) that is constructed by estimated control parameters \( \xi \).

(c) The adjoint problem is, then, solved by using the solution of the state problem in the previous step.

(d) The gradient of the objective functional, \( \nabla_{\xi} L \), is evaluated.
Finally, the gradient-based minimization scheme updates the estimated control parameters \( \xi \) via the conjugate-gradient method and the Newton’s method. The conjugate-gradient method determines the best search direction, and the Newton’s method determines an optimal step length.

The numerical optimizer repeats the above steps (b) to (e) and iteratively solve for the control parameters that satisfy the vanishing control equation. A set of these steps is counted as an inversion iteration. The detailed procedure of the numerical optimizer is summarized in Algorithm 1.

**Algorithm 1 Minimization Algorithm**

1: Set \( \text{TOL}=10^{-10} \)
2: Build \( M \) and \( K \) matrices for forward and adjoint wave solvers.
3: Compute \( u_m \) by using a forward wave solver with target traction \( F(x,t) \).
4: Set an iteration index \( s = 1 \) and initial control parameters \( (\xi_s)^{(1)} = 0 \) and compute \( \mathcal{L}_s^{(1)} \).
5: \( \text{while} \ ( \mathcal{L}_s^{(s+1)} > \text{TOL} \times \mathcal{L}_s^{(1)} \text{ and } s < 10^3 ) \text{ do} \)
6: Compute \( u(x,y,t) \) by solving the discrete form of the forward problem using \( \xi_s \).
7: Solve the discrete form of the adjoint problem.
8: Compute the components of a gradient vector, \( g_s = \nabla_\xi \mathcal{L}_s^{(s)} \).
9: Compute an optimal search direction \( d_s \) by using the conjugate-gradient scheme.
10: Compute an optimal step length \( h_s \) by using the Newton’s method.
11: Update estimated control parameters as \( \xi_s^{(s+1)} = \xi_s^{(s)} + d_s h_s \) and compute \( \mathcal{L}_s^{(s+1)} \).
12: \( s \leftarrow s + 1 \)
13: \text{end while}

### 2.4.1 Conjugate gradient

In every \( s \)-th inversion iteration, the gradient vector is computed as \( g_s = \nabla_\xi \mathcal{L}_s^{(s)} \). By using \( g_s \), the search direction \( d_s \) is computed by using the conjugate-gradient scheme [49]:

\[
\begin{align*}
    d_s &= -g_s, & (s = 0 \text{ and every } m \text{ (e.g., } m = 5)), \\
    d_s &= -g_s + \frac{g_s \cdot g_s}{g_{(s-1)} \cdot g_{(s-1)}} d_{(s-1)}, & (s \geq 1).
\end{align*}
\]  

(2.78)

The \( g_s \) is reset to be equal to \(-g_s\) at every \( m \) inversion iteration in order to eliminate the progressively-accumulated error in the search direction [9]. In the presented numerical experiments, we used \( m = 5 \).
2.4.2 Adaptively-calculated regularization factor

The value of the regularization factor $R$ in the above gradient (2.35) determines the extent, to which the penalty is imposed on the oscillation of the spatial and temporal variation of $F(x,t)$. If $R$ is too large, the estimated traction profile may remain too smooth. If $R$ is too small, the numerical optimizer could suffer from the solution multiplicity. To determine the optimal value of $R$ during the inversion process, this work adopts the regularization factor continuation scheme [9]. To this end, we decompose $\mathcal{L}$ into $\mathcal{L}_m$ and $\mathcal{R}^{TN} = R\mathcal{L}_R$, which are defined as:

\[
\mathcal{L}_m = \int_0^T \sum_{i=1}^{N_i} (u_{mi} - u_i)^2 \, dt,
\]

\[
\mathcal{L}_R = \frac{1}{2} \int_0^T \int_{\Gamma_b} \left( \frac{\partial F(x,t)}{\partial x} \right)^2 + \left( \frac{\partial F(x,t)}{\partial t} \right)^2 \, d\Gamma \, dt.
\]  

(2.79)

Accordingly, $\nabla_{(\xi=F_{kj})}\mathcal{L}$ in (2.35) can be decomposed into the following two in the OTD approach:

\[
\nabla_{(\xi=F_{kj})}\mathcal{L}_m = - \int_0^T \int_{\Gamma_b} \lambda \Phi_k(x) \phi_j(t) \, d\Gamma \, dt,
\]

\[
R\nabla_{(\xi=F_{kj})}\mathcal{L}_R = -R \int_0^T \int_{\Gamma_b} \left( \frac{\partial^2 F(x,t)}{\partial x^2} + \frac{\partial^2 F(x,t)}{\partial t^2} \right) \Phi_k(x) \phi_j(t) \, d\Gamma \, dt.
\]  

(2.81)

or the following two in the DTO approach: $\nabla_{(\xi=F_{kj})}\hat{\mathcal{L}}_m$ and $R\nabla_{(\xi=F_{kj})}\hat{\mathcal{L}}_R$, which are the components of the vectors, respectively, $-\hat{\mathbf{A}}$ and $R\mathbf{R}\hat{\mathbf{F}}$ in (2.71), at their rows corresponding to $\xi = F_{kj}$. Here, Kang and Kallivokas [9] suggested imposing the following inequality:

\[
R|\nabla_{\xi}\mathcal{L}_R| < |\nabla_{\xi}\mathcal{L}_m|, \quad \text{or} \quad R < \frac{|\nabla_{\xi}\mathcal{L}_m|}{|\nabla_{\xi}\mathcal{L}_R|}.
\]  

(2.82)

Thus, in each inversion iteration, $R$ can be set as:

\[
R = \frac{|\nabla_{\xi}\mathcal{L}_m|}{|\nabla_{\xi}\mathcal{L}_R|},
\]  

(2.83)
where \( I_R \) denotes the regularization intensity factor. Since Kang and Kallivokas [9] heuristically found that the value of \( I_R \) should be 0 < \( I_R < 0.5 \) for the material inversion, we tested the performance of the presented inversion with respect to \( I_R \) in the numerical experiments as well.

### 2.4.3 Updating the estimated control parameters

Starting with an initial guess for the control parameter vector \( \xi \), which is comprised of \( F_{kj} \), the estimate of \( \xi \) can be updated iteratively as:

\[
\xi_{(s+1)} = \xi_{(s)} + h_{(s)} d_{(s)}, \tag{2.84}
\]

where \( h_{(s)} \) is a scalar-valued step size and \( d_{(s)} \) is a search-direction vector computed by the conjugate-gradient scheme, and \( s \) is the iteration index in the optimization process. For each iteration, \( s \), the Newton’s method [8] is used to determine the optimal step size, \( h_{(s)} \). Namely, in each \( s \)-th iteration in the numerical optimization process, there are up to 4 sub-iterations (\( r \) is the sub-iteration index). We begin from an initially estimated \( h_{s(r=1)} \), and, for the next sub-iteration of \( (r+1) \) up to \( r \) of 4, we update \( h_{s(r+1)} \) as the following:

\[
h_{s(r+1)} = h_{sr} - \left( \frac{\mathcal{L}'(h_{sr})_{(s+1)}}{\mathcal{L}''(h_{sr})_{(s+1)}} \right) = h_{sr} - \left( \frac{\mathcal{L}(h_{sr} + \eta)_{(s+1)} - \mathcal{L}(h_{sr} - \eta)_{(s+1)}}{2\eta} \right)
\]

\[
= \frac{\mathcal{L}(h_{sr} + \eta)_{(s+1)} - 2\mathcal{L}(h_{sr})_{(s+1)} + \mathcal{L}(h_{sr} - \eta)_{(s+1)}}{\eta^2} \tag{2.85}
\]

where \( \mathcal{L}(h_{sr})_{(s+1)} \) is the objective functional given the updated \( \xi \) in (2.84) using \( h_{sr} \). For each sub-iteration, \( r \), values for the first and second derivatives of the objective functional with respect to \( h_s \)—i.e., \( \mathcal{L}'(h_{sr})_{(s+1)} \) and \( \mathcal{L}''(h_{sr})_{(s+1)} \)—are determined numerically via central difference approximations. In the very right hand side term of (2.85), \( \mathcal{L}(h_{sr} \pm \eta)_{(s+1)} \) is \( \mathcal{L}_{(s+1)} \) evaluated when \( \xi_{(s+1)} = \xi_s + (h_{sr} \pm \eta)d_s \).
2.5 Numerical Experiments

This section shows a set of numerical examples, investigating the performance of the presented inverse modeling with respect to various factors. In all the examples, we consider a square-shaped solid domain, of which extent is 60 m × 60 m. To avoid an inverse crime, we compute the synthetic measured data $u_m$ using an element size set as 0.5 m, while an element size of 1 m is used in the FEM solvers for obtaining state and adjoint solutions in the computational domain. The same time step of 0.001 s is used in the forward and inversion problems. The solid has a uniform mass density $\rho$ of 1000 kg/m$^3$.

Two targeted dynamic tractions, $F_1(x,t)$ and $F_2(x,t)$, are considered in this section. The targeted $F_1(x,t)$, of which amplitude changes over space and time, is shown in Fig. 2.6(a). Its peak amplitude is 200 N/m$^2$, and the total observation duration $T_1$ is 1.5 s in the inversion simulation of $F_1(x,t)$. The time-dependent value of the targeted $F_1(x,t)$ at any specific value of $x$ is a Ricker wavelet (for instance, see $F_1(30,t)$ in Fig. 2.3(a)) with its central frequency of 10 Hz (see the frequency contents of $F_1(30,t)$ in Fig. 2.3(b)).

The targeted $F_2(x,t)$ is shown in Fig. 2.19(a), and its amplitude changes over space and time. Fig. 2.3(c) shows the time-dependent value of the targeted $F_2(30,t)$ and Fig. 2.3(d) shows its frequency contents. The signal is a part of a recorded ground motion signal during the 1994 Northridge earthquake from the Pacific Earthquake Engineering Research Center (PEER) ground motion database [50]. The total observation duration $T_2$ is 6 s for the inversion simulation to identify the targeted $F_2(30,t)$.

For the inversion process, we discretize estimated $F_1(x,t)$ by using 91,500 control parameters—i.e., 61 (over space) × 1500 (over time)—and $F_2(x,t)$ by using 366,000 control parameters—i.e., 61 (over space) × 6000 (over time). The temporal and spatial intervals for the discretization are 0.001 s and 1 m, respectively, for both forces. Initially-estimated values of all the control parameters are zero. Our numerical optimizer iteratively updates the values of all the control parameters by using the presented inverse modeling procedure.
Figure 2.3: (a) The time signal of $F_1(x = 30, t)$; (b) the amplitude of Fourier Transform of $F_1(x = 30, t)$; (c) the time signal of $F_2(x = 30, t)$; and (d) the FFT of $F_2(x = 30, t)$. 
In what follows, six examples of numerical experiments are presented. The first example is focused on the performance of inverting for $F_1(x,t)$ with respect to the complexity of the material profile in the domain. To this end, we consider the following material profiles:

- **Material profile 1**: A homogeneous solid with the shear wave speed of $V_s = 250$ m/s,

- **Material profile 2**: A 2-layered solid with 1 inclusion as in Fig. 2.4(a) with shear wave speeds of $V_{s1} = 300$ m/s, $V_{s2} = 350$ m/s, and $V_{s3} = 200$ m/s,

- **Material profile 3**: A 3-layered solid with 3 inclusions as in Fig. 2.4(b) with shear wave speeds of $V_{s1} = 400$ m/s, $V_{s2} = 450$ m/s, $V_{s3} = 300$ m/s, $V_{s4} = 350$ m/s, $V_{s5} = 200$ m/s, and $V_{s6} = 250$ m/s.

![Figure 2.4: Heterogeneous solids: (a) Material profile 2; and (b) Material profile 3.](image)

The second example compares the performance of identifying $F_1(x,t)$ by using the OTD method versus the DTO method. The third example tests the inversion performance of identifying $F_1(x,t)$ with respect to the number of sensors on the top surface of the domain. The fourth example examines the convergence of the estimated $F_1(x,t)$ into the target with respect to the regularization
intensity factor, $I_R$. The fifth example is focused on the performance of inverting for $F_1(x,t)$ with respect to the noise level. Lastly, the sixth example shows the capability of our inverse modeling to reconstruct $F_2(x,t)$, which is a realistic seismic signal as opposed to $F_1(x,t)$. Each example considers the result data of multiple cases, of which input parameters are summarized in Table 2.1.

For the sake of assessing the accuracy to reconstruct $F(x,t)$ in the numerical results, the following error norm between estimated $F(x,t)$ and its target is used:

$$
\mathcal{E} = \frac{\int_0^T \int_{\Omega} |F(x,t)_{\text{target}} - F(x,t)_{\text{estimate}}|^2 d\Omega dt}{\int_0^T \int_{\Omega} |F(x,t)_{\text{target}}|^2 d\Omega dt} \times 100[\%].
$$  \hspace{2cm} (2.86)

<table>
<thead>
<tr>
<th>Case number</th>
<th>Material profile</th>
<th>Force profile</th>
<th>Approach</th>
<th># of sensors</th>
<th>$I_R$</th>
<th>Noise level [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>1</td>
<td>OTD</td>
<td>30</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>1</td>
<td>OTD</td>
<td>30</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>3</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>30</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>4</td>
<td>3</td>
<td>1</td>
<td>DTO</td>
<td>30</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>5</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>6</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>10</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>7</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>5</td>
<td>0.5</td>
<td>0</td>
</tr>
<tr>
<td>8</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>1.0</td>
<td>0</td>
</tr>
<tr>
<td>9</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>10</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.01</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.0</td>
<td>1</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.0</td>
<td>2</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.0</td>
<td>3</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.01</td>
<td>2</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.1</td>
<td>2</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>1</td>
<td>OTD</td>
<td>15</td>
<td>0.5</td>
<td>2</td>
</tr>
<tr>
<td>17</td>
<td>3</td>
<td>2</td>
<td>OTD</td>
<td>15</td>
<td>0.0</td>
<td>0</td>
</tr>
</tbody>
</table>

Before our parametric studies on the inversion performance, we verify the theoretical derivation of the adjoint and control equations and the numerical implementation. That is, the gradients
obtained by our semi-analytical OTD and DTO approaches are compared with that from the finite difference (FD) approach. In this verification, we used the targeted $F_1(x,t)$, a heterogeneous domain of the material profile 3, shown in Fig. 2.4(b), only one sensor in the center of $\Gamma_t$, $I_R$ of 0, and the noise level of 0%. In order to reduce the computational cost of the FD approach, the spatial distribution of $F_1(x,t)$ is set to be uniform, and, thus, only the temporal variation of $F_1(x,t)$ is considered. Fig. 2.5 shows excellent agreement among the normalized gradients (i.e., $\nabla_\xi \mathcal{L} / |\nabla_\xi \mathcal{L}|$) that are calculated by using the OTD approach, the DTO approach, and the FD approach, respectively, at the first inversion iteration. Thus, our theoretical derivation and numerical implementation of the presented inverse modeling are trustable.

Figure 2.5: Comparison between the gradients generated by the OTD approach, DTO approach and the FD approximation.
2.5.1 Example 1: Investigating the inversion performance with respect to the material profile complexity.

In this example, we test the performance of the presented inverse modeling with respect to the complexity of the material profile by using the results of Cases 1 to 3. We used the OTD approach, 30 sensors, $I_R$ of 0.5, and the noise level of 0% in these Cases.

Fig. 2.6(b,c,d) show reconstructed dynamic traction for Cases 1, 2, and 3, respectively. Figs. 2.7 and 2.8 show that the values of $\mathcal{L}$ and $\mathcal{E}$, in general, decrease as the number of iterations increases. Fig. 2.7 depicts that $\mathcal{L}$ shows a sawtooth behavior over iterations. We suggest that it occurs because of the penalty that is imposed by the regularization term on the derivative of $F_1(x,t)$. Namely, as seen in the later Examples 4 and 6, when $I_R$ is equal to zero, the sawtooth behavior of $\mathcal{L}$ does not occur. Besides, Fig. 2.8 shows that the more complex the material profile is, the higher terminal value of $\mathcal{E}$ is obtained. Fig. 2.9 shows that the wave responses of $u_m$ due to the targeted $F_1(x,t)$ match those of $u$ due to the reconstructed one at two sensors in Cases 1-3. It implies that our numerical minimizer is very effective in minimizing the misfit: there is a very small difference, of the scale of $10^{-13}$ to $10^{-12}$, among the terminal values of $\mathcal{L}$ in Fig. 2.7 for Cases 1 to 3. Even though $u$ match $u_m$ at the end of the inversion simulation, our optimizer results in a higher terminal value of $\mathcal{E}$ for a more complex material profile. To explain this aspect, we note that Lloyd and Jeong [8] reported that a more heterogeneous background solid leads to a larger terminal value of an error between a targeted moving, body force-typed wave source function and its estimated counterpart in a 1D solid setting. It is because, the more heterogeneous the material property of a domain is, the waves reverberate more inside the domain. That is, because of the reverberation, as seen in Fig. 2.6, the reconstructed traction has a stronger noise-like behavior, leading to a higher final value of $\mathcal{E}$ when a more heterogeneous material profile is used. Thus, in a more heterogeneous solid setting, the inversion solver is less likely to converge towards a targeted traction function.

Fig. 2.10 shows the snapshots of the wave responses in the entire computational domain, induced by (a) the targeted $F_1(x,t)$ and (b) its reconstructed counterpart in Case 3. Both responses
Figure 2.6: (a) Target and (b-d) Reconstructed $F_1(x,t)$, in N/m$^2$, for Cases 1-3 at the 1000th iteration. Horizontal and vertical axis in the contour plot represent the numbering of the discretized points over space ($x$) and time ($t$) of the distribution of $F_1(x,t)$, respectively.
Figure 2.7: Example 1 - Objective function, $L$, versus iterations with respect to the material profiles.

Figure 2.8: Example 1 - After 1000 iterations, the final value of $\varepsilon$ is 11.04% in Case 1, 16.87% in Case 2, and 25.02% in Case 3.
are in excellent agreement with each other even though the terminal value of $\mathcal{E}$ in Case 3 is the largest among the Cases 1-3. Thus, the presented dynamic-input inversion algorithm could be further developed for reconstructing seismic input motions and, then, “replaying” the corresponding wave responses in a truncated computational domain during seismic events.

Figure 2.9: $u_m$ and $u$, at sensors placed on the top surface. (a-b) measured at $x = 20$ m and $x = 40$ m, respectively, for Case 1. (c-d) measured at $x = 20$ m and $x = 40$ m, respectively, for Case 2. (e-f) measured at $x = 10$ m and $x = 20$ m, respectively, for Case 3.
Figure 2.10: Wave responses, $u(x,y,t)$ [m], at 1.5 seconds due to (a) target traction and (b) reconstructed traction in Case 3, and (c) the difference between them.
2.5.2 Example 2: Comparison between the inversion performances of the OTD and DTO approaches

This example compares the performance of the OTD approach with that of the DTO counterpart by using the results of Cases 3 and 4, in both of which we reconstruct the targeted traction $F_1(x,t)$ by using the material profile 3, $I_R$ of 0.5, and the noise level of 0%. Fig. 2.11 shows the targeted $F_1(x,t)$ and its reconstructed counterparts obtained by using the OTD and DTO approaches, in Cases 3 and 4, respectively. We could not visually notice the difference between the two reconstructed ones in Fig. 2.11. Thus, we suggest that there is no difference between the final reconstructed traction function obtained by using the OTD approach and that by using the DTO approach in this 2D SH wave work. Fig. 2.12 also shows that the terminal values of $L$ and $E$, when using the OTD approach, are in excellent agreement with those using the DTO counterpart, with a difference of 0.33% in $E$. However, a sawtooth behavior of $L$ occurs more when we use the DTO approach than the OTD approach. As mentioned earlier, the sawtooth behavior occurs because of the regularization. Namely, as shown in (2.76) and (2.77), the gradient of the regularization term in the OTD approach is implemented differently from the DTO counterpart so that their corresponding behaviors of $L$ differ from each other.

2.5.3 Example 3: Investigating the inversion performance with respect to the number of sensors

In this example, we study the inversion performance with respect to the number of sensors on the top surface. We considered Cases 3, 5, 6, and 7, which use 30, 15, 10, and 5 sensors, respectively, and the material profile 3, $I_R$ of 0.5, and the noise level of 0%. Fig. 2.13 shows that, although increasing the number of sensors decreases the terminal value of $E$ of the inversion, using 5, 10, 15, and 30 sensors gives rise to the terminal values of $E$ of the almost same magnitude, i.e., 25 to 26% with a slight difference of only up to 1% from each other. Thus, we suggest that when the sampling rate of the measurement is equal to the timestep for discretizing $F_1(x,t)$, the ratio of the
Figure 2.11: Wave responses, $u(x,y,t)[\text{m}]$, at 1.5 seconds due to (a) target traction and (b,c) reconstructed traction for the OTD and DTO approach, respectively. (d,e) The difference in wave responses between target and OTD approach and target and DTO approach, respectively.
Figure 2.12: Example 2 - $L$ and $\mathcal{E}$ with respect to the inverse approach.
size of measurement data to the number of the control parameters can be as small as 1:12—e.g., (5 sensors × 1500 timesteps):(61 nodes × 1500 timesteps)—in the presented example.

2.5.4 Example 4: Investigating the inversion performance with respect to \( I_R \)

This example studies the accuracy of the inversion with respect to the regularization intensity factor, \( I_R \). We used 15 sensors on the top surface, the material profile 3, and the noise level of 0% for all the cases in this example. In addition to the Case 5 in Example 3, where \( I_R \) of 0.5 is used, we considered Cases 8-11, which use \( I_R \) of 1.0, 0.1, 0.01, and 0.0, respectively.

As can be seen in Fig. 2.14, the terminal value of \( \mathcal{L} \) for a large value of \( I_R \) (e.g., \( I_R = 1.0 \)) is larger than those for smaller values of \( I_R \). We also note that, when \( I_R \) is 0.0, the sawtooth behavior of \( \mathcal{L} \) does not occur in Fig. 2.14. Furthermore, Fig. 2.15 shows that the terminal value of \( \mathcal{E} \) for \( I_R \) of 1.0 is 54.9 % and is about twice higher than those (about 25 %) for \( 0 \leq I_R \leq 0.5 \).

The inversion performance, in terms of \( \mathcal{E} \) over the iterations, is relatively unstable for \( I_R \) of 1.0, while it is stable for \( 0 \leq I_R \leq 0.5 \). That is, using \( I_R \) of 1.0 increases \( \mathcal{E} \) after the 180-th iteration. Fig. 2.16 compares the final estimated traction function, when \( I_R \) is 1.0, with that for \( I_R \) of 0.0. Fig. 2.16 shows that the noise-like behavior is more severe in the reconstructed traction for \( I_R \) of 1.0 than that for \( I_R \) of 0.0. We also note that when \( I_R \) of 0.0 was used, \( \mathcal{E} \) is the smallest (24.65%) in this example as shown in Fig. 2.15.

We discuss why using \( I_R \) of 0.0 leads to as a small terminal value of \( \mathcal{E} \) as \( I_R \) greater than 0.0 in the following. As discussed in the previous 1D work of the seismic-input inversion [7], the proposed dynamic-traction inversion is naturally equipped with the smoothing effect even without using the TN regularization. Namely, the FEM solver naturally filters out high frequencies of the estimated traction function and smooths its temporal variation even without the regularization method. Similarly, it filters out the high-wavelength content of the spatial variation of an estimated traction function. Because of such inherent low-pass filtering of the FEM solver, as shown in our
Figure 2.13: Example 3 - Error, $\mathcal{E}$, versus iterations with respect to the number of sensors.
presented numerical results, even when TN regularization is not used, the inversion solver smooths an estimated traction function.

![Figure 2.14: Example 4 - Objective functional, $\mathcal{L}$, versus iterations with respect to $I_R$.](image)

**2.5.5 Example 5: Investigating the inversion performance with respect to the noise level**

In this example, we focus on examining the performance of inverting $F_1(x,t)$ with respect to the noise level of random noise that is added to $u_m$ prior to the inversion. We used the OTD approach, the material profile 3, 15 sensors, $I_R$ of 0.0, and examined the noise level of 0%, 1%, 2%, and 3%, which correspond to Cases 11-14, respectively. Fig. 2.17 shows that, the larger the noise level is, the larger the terminal values of $\mathcal{L}$ and $E$ are obtained when no regularization is used.

Fig. 2.18 shows the inversion performance with respect to $I_R$ (Cases 15-17 in addition to the Case 13) when we use the material profile 3, 15 sensors, and the noise level of 2%. We note that $I_R$ of 0.01, 0.1, and 0.5 did not make any difference in the terminal value of $E$ compare to that
Figure 2.15: Example 4 - Error, $\varepsilon$, versus iterations with respect to $I_R$. 

(a) Overall scale

(b) Close-up scale
for $I_R$ of 0.0. Thus, we suggest that using the TN regularization does not improve the inversion performance in the presented work when $u_m$ contains noises.

### 2.5.6 Example 6: Examining the feasibility of the presented inverse modeling to reconstruct a realistic seismic signal $F_2(x,t)$

In this example, we focus on examining the feasibility of inverting for a realistic seismic signal $F_2(x,t)$. We used the OTD approach, the material profile 3, 15 sensors, $I_R$ of 0.0, and the noise level of 0%. Fig. 2.19 shows the excellent agreement between the targeted and reconstructed dynamic tractions, $F_2(x,t)$, in Case 18. Fig. 2.20 and 2.21 show the values of $\mathcal{L}$ and $\mathcal{E}$, respectively, over iterations, and $\mathcal{L}$ decreases without the sawtooth behavior because $I_R$ of 0.0 is used. Overall, $F_2(x,t)$ has much lower frequency content than $F_1(x,t)$ so that the wave responses induced by $F_2(x,t)$ are less complex than those by $F_1(x,t)$ (see the wave responses in Fig. 2.22 and compare them with those in Fig. 2.10). Thus, our minimizer suffers from solution multiplicity less severely when it identifies $F_2(x,t)$ than $F_1(x,t)$. Accordingly, the terminal value of $\mathcal{E}$, 3.86%, for reconstructing $F_2(x,t)$ in Case 18 is much smaller than its counterpart, 29.76%, of reconstructing $F_1(x,t)$ in Case 11. Fig. 2.22 shows the snapshots of the wave responses in the entire computational...
Figure 2.17: Example 5 - $\mathcal{L}$ and $\mathcal{E}$ with respect to the noise level.
Figure 2.18: Example 5 - $\mathcal{L}$ and $\varepsilon$ for 2% of noise with respect of $I_R$. 

(a) Misfit, $\mathcal{L}$, versus iterations

(b) Error, $\varepsilon$, versus iterations
domain, induced by (a) the targeted $F_2(x,t)$ and (b) its reconstructed counterpart in Case 18. In general, both responses are in excellent agreement with each other.

Figure 2.19: Example 6 - (a) Target and (b) Reconstructed $F_2(x,t)$, in N/m², for Case 18 at the 6000-th iteration.
Figure 2.20: Example 6 - Objective function, $\mathcal{L}$, decreases without the sawtooth behavior because $I_R$ of 0.0 is used.

Figure 2.21: Example 6 - After 6000 iterations, the final value of $\varepsilon$ is 3.85% in Case 18.
Figure 2.22: Wave responses, $u(x,y,t)$ [m], at 6.0 seconds due to (a) target load and (b) reconstructed load in Case 18, and (c) the difference between them.
2.6 Summary

We present the mathematical modeling and numerical implementation of a new inversion process for identifying the spatial and temporal distributions of dynamic traction applied on a boundary of a 2D solid domain with SH scalar wave motions. We tackle the inverse problem by using a gradient-based minimization scheme. The gradient of an objective functional is evaluated semi-analytically by using the adjoint solution. We present both OTD and DTO methods, each of which resolves the adjoint problem differently from each other.

Numerical results show the following findings of the performance of this new inversion method. First, the complexity of the material profile in a domain increases the error between the reconstructed traction and its target. The more heterogeneous the material property of a domain is, the waves reverberate more inside the domain. Because of the reverberation, the reconstructed traction has a stronger noise-like behavior, leading to a higher terminal value of $E$ when a more heterogeneous material profile is used. Thus, in a more heterogeneous solid setting, the inversion solver is less likely to converge towards a targeted traction function. Second, the OTD and DTO methods lead to the same inversion results, but the distribution of $L$ over the iterations shows a more significant sawtooth behavior when we use the DTO method than the OTD method. Third, when the sampling rate of the measurement is equal to the timestep for discretizing $F(x,t)$, the ratio of the size of measurement data to the number of the control parameters can be as small as 1:12 in the presented work. Fourth, the regularization intensity $I_R$ should not be too large: for instance, $I_R$ is recommended to be smaller than and equal to 0.5. We also note that the terminal value of $E$, when $I_R$ of 0 is used, is as small as those when using $0.01 \leq I_R \leq 0.5$. Thus, it is acceptable to tackle the presented inverse modeling of dynamic tractions without the regularization. Fifth, the terminal values of $L$ and $E$ increase as the noise of a larger level is added to $u_m$, and using the TN regularization does not improve the inversion performance when noise is added to $u_m$. Sixth, our minimizer suffers from solution multiplicity less when it identifies dynamic traction of lower frequency content than that of higher frequency content.
As shown in the numerical results, the wave responses in the entire computational domain, induced by the targeted traction and the reconstructed one, are in excellent agreement with each other in the presented highly-reverberating domain. Thus, if the presented dynamic-input inversion algorithm is extended into realistic 3D settings (see below for the details of the extension), it could allow engineers to reconstruct incident seismic motions and, then, to replay the wave responses in a 3D truncated domain.
CHAPTER 3

FULL-WAVEFORM INVERSION OF SEISMIC INPUT MOTIONS IN A DOMAIN TRUNCATED BY USING ABSORBING BOUNDARY CONDITIONS

3.1 Problem Definition

This Chapter explores a new full-waveform inversion method for reconstructing the spatial and temporal distributions of dynamic traction in a domain that is truncated by using a wave-absorbing boundary condition (WABC) from sparsely-measured wave response data (see Fig. 3.1).

Figure 3.1: The problem configuration using WABC
The governing equation

The governing equation used in this problem is the same as previous chapter that represents the SH wave propagation in the undamped solid domain:

$$\nabla \cdot (G \nabla u) - \rho \frac{\partial^2 u}{\partial t^2} = 0, \quad \text{on } \Omega \times J,$$

(3.1)

The solid is subject to a traction-free condition on the top surface ($\Gamma_t$):

$$\frac{\partial u}{\partial y}(x,0,t) = 0, \quad 0 \leq x \leq L.$$

(3.2)

The absorbing boundary conditions are presented on the bottom ($\Gamma_b$), left ($\Gamma_l$) and right ($\Gamma_r$) boundaries:

$$\frac{\partial u}{\partial y}(x,D,t) = \frac{F(x,D,t)}{G} - \frac{1}{V} \frac{\partial u}{\partial t}, \quad 0 \leq x \leq L,$$

(3.3)

$$\frac{\partial u}{\partial x}(0,y,t) = \frac{F(0,y,t)}{G} - \frac{1}{V} \frac{\partial u}{\partial t}, \quad D \leq y \leq 0,$$

(3.4)

$$\frac{\partial u}{\partial x}(L,y,t) = -\frac{1}{V} \frac{\partial u}{\partial t}, \quad D \leq y \leq 0,$$

(3.5)

where $D$ is the $y$-coordinate of $\Gamma_b$, $L$ is the $x$-coordinate of $\Gamma_r$, $V(x,y)$ is the shear wave speed of the medium. We note that $F(x,y,t)$ denotes the dynamic traction of shear stress applied in the out-of-plane direction on the boundaries, and $F(x,y,t)$ is used for modeling the incoming seismic wavefield arriving at the WABC boundary $^1$:

$$F(x,y,t) = \frac{2G}{V} \frac{\partial u^I}{\partial t},$$

(3.6)

where $u^I$ denotes the incident seismic wavefield.

$^1$This modeling approach is simpler than Bielak’s Domain Reduction Method (DRM) [51]. In this work, we are testing our inverse algorithm before we test it with the DRM.
The governing wave physics is also subject to zero initial-value conditions:

\[ u(x, y, 0) = 0, \quad (3.7) \]
\[ \frac{\partial u}{\partial t}(x, y, 0) = 0. \quad (3.8) \]

### 3.2 Forward Wave Modeling

The weak form of the state problem is cast in order to find \( u \):

\[
\int_{\Omega} \nabla v \cdot (G \nabla u) \, d\Omega + v \left( \int_{\Gamma_r} \frac{G}{V} \frac{\partial u}{\partial t} \, d\Gamma + \int_{\Gamma_l} \frac{G}{V} \frac{\partial u}{\partial t} \, d\Gamma + \int_{\Gamma_b} \frac{G}{V} \frac{\partial u}{\partial t} \, d\Gamma \right) + \int_{\Omega} v \rho \frac{\partial^2 u}{\partial t^2} \, d\Omega = \int_{\Gamma_b} v F(x, y, t) \, d\Gamma + \int_{\Gamma_l} v F(x, y, t) \, d\Gamma. \quad (3.9)
\]

Approximating the test and trial functions, \( v(x, y) \) and \( u(x, y, t) \), (3.9) reduces to:

\[
M \ddot{u}(t) + C \dot{u}(t) + K u(t) = F(t), \quad (3.10)
\]

where \( (\cdot) \) and \( (\cdot) \) denotes, respectively, the first and second-order derivative of its subtended variable with respect to \( t \); \( M \) denotes a global mass matrix; \( C \) denotes a global dumping matrix; \( K \) denotes a global stiffness matrix; \( F \) denotes a global force vector; and \( u(t) \) is the solution vector.

The vectors and matrices are defined as:

\[
K = \int_{\Omega} G \left( \frac{\partial \psi}{\partial x} \frac{\partial \psi^T}{\partial x} + \frac{\partial \psi}{\partial y} \frac{\partial \psi^T}{\partial y} \right) \, d\Omega,
\]
\[
C = \int_{\Gamma_r} \frac{G}{V} \psi \psi^T \, d\Gamma + \int_{\Gamma_l} \frac{G}{V} \psi \psi^T \, d\Gamma + \int_{\Gamma_b} \frac{G}{V} \psi \psi^T \, d\Gamma,
\]
\[
M = \int_{\Omega} \rho \psi \psi^T \, d\Omega,
\]
\[
F = \int_{\Gamma_b} \psi F \, d\Gamma + \int_{\Gamma_l} \psi F \, d\Gamma. \quad (3.11)
\]
Solving (3.9) in every discrete $i$-th time step as:

$$\mathbf{M\ddot{u}}_i + \mathbf{C\dot{u}}_i + \mathbf{Ku}_i = \mathbf{F}_i.$$  \hfill (3.12)

Applying the initial value-conditions, the solution vector at the initial time step is obtained by solving the following:

$$\mathbf{M\ddot{u}}_0 = \mathbf{F}_0,$$  \hfill (3.13)

Applying Newmark time integration, the solution vector of a next time step is related to that of a previous time step as:

$$\dot{\mathbf{u}}_i = \frac{2}{\Delta t}(\mathbf{u}_i - \mathbf{u}_{i-1}) - \mathbf{\dot{u}}_{i-1},$$  \hfill (3.14)

and

$$\ddot{\mathbf{u}}_i = \frac{4}{(\Delta t)^2}(\mathbf{u}_i - \mathbf{u}_{i-1}) - \frac{4}{\Delta t}\dot{\mathbf{u}}_{i-1} - \ddot{\mathbf{u}}_{i-1},$$  \hfill (3.15)

where $i$ denotes the $i$-th time step, and $\Delta t$ denotes the size of a time step.

By plugging (3.14) and (3.15) into (3.12), it turns into the following:

$$\left[ \frac{4}{(\Delta t)^2}\mathbf{M} + \frac{2}{\Delta t}\mathbf{C} + \mathbf{K} \right] \mathbf{u}_i = \mathbf{F}_i + \mathbf{M} \left( \frac{4}{(\Delta t)^2}\mathbf{u}_{i-1} + \frac{4}{\Delta t}\dot{\mathbf{u}}_{i-1} + \ddot{\mathbf{u}}_{i-1} \right) + \mathbf{C} \left( \frac{2}{\Delta t}\mathbf{u}_{i-1} + \dot{\mathbf{u}}_{i-1} \right)$$  \hfill (3.16)

By using (3.16), this work solves for $\mathbf{u}_i$. Then, the values of $\dot{\mathbf{u}}_i$ and $\ddot{\mathbf{u}}_i$ can be updated by using $\mathbf{u}_i$ and, respectively, Eqs. (3.14) and (3.15). Equations (3.13) to (3.16) can be cast in the following compact form:

$$\mathbf{Q}\hat{\mathbf{u}} = \hat{\mathbf{F}}.$$  \hfill (3.17)
where the vectors $\hat{u}$ and $\hat{F}$ are defined as:

$$
\hat{u} = \begin{bmatrix}
  u_0 \\
  \dot{u}_0 \\
  \ddot{u}_0 \\
  \vdots \\
  u_N \\
  \dot{u}_N \\
  \ddot{u}_N \\
  \hat{u}_N
\end{bmatrix}, \quad \hat{F} = \begin{bmatrix}
  0 \\
  0 \\
  F_0 \\
  \vdots \\
  F_N \\
  0 \\
  0 \\
  0
\end{bmatrix},
$$

(3.18)

and the matrix $Q$ is:

$$
Q = 
\begin{bmatrix}
  I & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  0 & I & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  K & C & M & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  L_1 & L_2 & L_3 & \text{Keff} & 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  a_1 I & I & 0 & -a_1 I & I & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  a_0 I & a_2 I & I & -a_0 I & 0 & I & \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & L_1 & L_2 & L_3 & \text{Keff} & 0 & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_1 I & I & 0 & -a_1 I & I & 0 \\
  0 & 0 & 0 & 0 & 0 & 0 & \cdots & a_0 I & a_2 I & I & -a_0 I & 0 & I
\end{bmatrix},
$$

(3.19)

where:

$$
\text{Keff} = a_0 M + a_1 C + K, \quad L_1 = -a_0 M - a_1 C, \quad L_2 = -a_2 M - C, \quad L_3 = -M
$$

(3.20)
In (3.19), the 1st and 2nd rows correspond to the initial conditions \( u_0 = 0 \) and \( \dot{u}_0 = 0 \); The 3rd row is equivalent to (3.13); The 4, 7, 10, ...-th row correspond to (3.16); The 5, 8, 11, ...-th rows are (3.14); The 6, 9, 12, ...-th rows are (3.15). Instead of the time marching scheme, by solving the matrix system of (3.17) "all at once", we could obtain the forward wave solutions. However, to save the computational cost, we opt for using the time marching scheme for solving (3.17).

3.3 Inverse Modeling

This section presents mathematical modeling for identifying the temporal and spatial distributions of an unknown dynamic traction \( F(x, y, t) \)—mimicking unknown incoming seismic wavefield—based on measured wave responses on the top surface of the solid. The properties of the system, such as the extents and the material properties of the solid, are assumed to be known. This work uses the discretize-then-optimize (DTO) approach. Namely, the objective functional is cast in a discrete form, and, then, the discrete optimality conditions are derived and implemented.

3.3.1 Parameterization of a dynamic input traction function

We define an estimated dynamic input traction function, \( P(\gamma, t) \), as the following:

\[
P(\gamma, t) = F(0, -\gamma, t), \quad 0 \leq \gamma < -D, \\
P(\gamma, t) = F(\gamma + D, D, t), \quad -D \leq \gamma \leq -D + L.
\]  (3.21)

We approximate \( P(\gamma, t) \) as:

\[
P(\gamma, t) = \sum_{k=1}^{N_y} \sum_{j=1}^{N_t} \Phi_k(\gamma) \phi_j(t) P_{kj},
\]  (3.22)

where \( \Phi_k(\gamma) \) denotes the \( k \)-th component of a vector of global basis functions used for the spatial discretization of \( P(\gamma, t) \); \( \phi_j(t) \) denotes the \( j \)-th component of a vector of global basis functions.
used for the temporal discretization of \( P(\gamma,t) \); \( P_{kj} \) denotes the discretized value of \( P(\gamma,t) \) at each discrete location \( \gamma_k \) and time \( t_j \); and \( N_\gamma \) and \( N_t \) denote the numbers of discretization points over space and time, respectively.

### 3.3.2 Discrete objective and Lagrangian functional

The discrete objective functional is defined as:

\[
\hat{L} = \frac{1}{2} (\hat{u} - \hat{u}_m)^T \hat{B} (\hat{u} - \hat{u}_m), \tag{3.23}
\]

To tackle the minimization problem, the following Lagrangian functional \( \mathcal{A} \) is built by imposing the discrete form of the forward wave problem, using a Lagrange multiplier vector \( \hat{\lambda} \), onto \( \hat{L} \):

\[
\mathcal{A} = \frac{1}{2} (\hat{u} - \hat{u}_m)^T \hat{B} (\hat{u} - \hat{u}_m) - \hat{\lambda}^T (\hat{Q}\hat{u} - \hat{F}). \tag{3.24}
\]

### 3.3.3 The first order optimality conditions

To identify unknown, target control parameters, the following optimality conditions must be satisfied:

\[
\frac{\partial \mathcal{A}}{\partial \hat{\lambda}} = 0 \quad \text{The first condition (state problem),} \tag{3.25}
\]

\[
\frac{\partial \mathcal{A}}{\partial \hat{u}} = 0 \quad \text{The second condition (adjoint problem),} \tag{3.26}
\]

\[
\frac{\partial \mathcal{A}}{\partial \hat{F}} = 0 \quad \text{The third condition (control problem).} \tag{3.27}
\]

The first condition, the vanishing derivative of \( \mathcal{A} \) with respect to \( \hat{\lambda} \), will be automatically satisfied when we solve the discrete forward problem (3.17), which is also referred to as the discrete state equation:

\[
\frac{\partial \mathcal{A}}{\partial \hat{\lambda}} = -\hat{Q}\hat{u} + \hat{F} = 0. \tag{3.28}
\]
As the second condition, the gradient of $\mathcal{A}$ with respect to $\hat{u}$ will be automatically satisfied when we solve the following discrete adjoint problem:

$$\frac{\partial \mathcal{A}}{\partial \hat{u}} = \frac{\partial \mathcal{L}}{\partial \hat{u}} - \frac{\partial (\hat{\lambda}^T Q \hat{u})}{\partial \hat{u}} = 0$$ \hspace{1cm} (3.29)

Part $a$ in (3.29) can be written as:

$$a = \frac{1}{2} \frac{\partial (\hat{u} - \hat{u}_m)^T}{\partial \hat{u}} \bar{B} (\hat{u} - \hat{u}_m) + \frac{1}{2} (\hat{u} - \hat{u}_m)^T \bar{B} \frac{\partial (\hat{u} - \hat{u}_m)}{\partial \hat{u}}$$

$$= \frac{1}{2} \frac{\partial (\hat{u} - \hat{u}_m)^T}{\partial \hat{u}} \bar{B} (\hat{u} - \hat{u}_m) + \frac{1}{2} \frac{\partial (\hat{u} - \hat{u}_m)^T}{\partial \hat{u}} \bar{B} (\hat{u} - \hat{u}_m)$$

$$= \bar{B} (\hat{u} - \hat{u}_m) \hspace{1cm} (3.30)$$

Part $b$ in (3.29) can be written as:

$$b = - \frac{\partial}{\partial \hat{u}} (\hat{\lambda}^T Q \hat{u}) = - \frac{\partial}{\partial \hat{u}} (\hat{\lambda}^T Q \hat{u})^{\text{scalar}}$$

$$= - \frac{\partial}{\partial \hat{u}} ((Q \hat{u})^T \lambda) = - \frac{\partial}{\partial \hat{u}} (\hat{u}^T Q^T \lambda)$$

$$= - Q^T \lambda \hspace{1cm} (3.31)$$

Due to (3.30) and (3.31), (3.29) can be written as:

$$\frac{\partial \mathcal{A}}{\partial \hat{u}} = - Q^T \hat{\lambda} + \bar{B} (\hat{u} - \hat{u}_m) = 0, \hspace{1cm} \text{adjoint equation} \hspace{1cm} (3.32)$$

Since the adjoint problem involves transpose of $Q$, we solve it by marching backwards in time. Namely, from the last two rows of (3.19), we obtain the final conditions:

$$\dot{\hat{\lambda}}_N = 0, \quad \hat{\lambda}_N = 0, \hspace{1cm} (3.33)$$
respectively; and the third row from the bottom yields:

\[ K_{\text{eff}}^T \lambda_N = \Delta t B (u_N - u_{mN}) + a_1 \dot{\lambda}_N + a_0 \ddot{\lambda}_N, \]  

(3.34)

which can be solved for \( \lambda_N \). For time steps \( i = N - 1, N - 2, \ldots, 1 \), we first update the following:

\[ \dot{\lambda}_i = M^T \lambda_{i+1} - \ddot{\lambda}_{i+1}, \]  

(3.35)

\[ \dot{\lambda}_i = (a_2 M^T + C^T) \dot{\lambda}_{i+1} - \ddot{\lambda}_{i+1} - a_2 \dddot{\lambda}_{i+1}. \]  

(3.36)

Then, solve the following:

\[ K_{\text{eff}}^T \lambda_i = \Delta t B (u_i - u_{mi}) + a_1 \dot{\lambda}_i + a_0 \ddot{\lambda}_i + (a_0 M^T + a_1 C^T) \dot{\lambda}_{i+1} + a_1 \dddot{\lambda}_{i+1} - a_0 \dddot{\lambda}_{i+1}. \]  

(3.37)

Finally, the first three rows of (3.19) result in the following equations:

\[ M^T \ddot{\lambda}_0 = M^T \lambda_1 - \ddot{\lambda}_1 \]  

(3.38)

\[ \dot{\lambda}_0 = -C^T \dot{\lambda}_0 + (a_2 M^T + C^T) \lambda_1 - \dot{\lambda}_1 - a_2 \dddot{\lambda}_1, \]  

(3.39)

\[ \lambda_0 = -K^T \dot{\lambda}_0 + (a_0 M^T + a_1 C^T) \lambda_1 - a_1 \dot{\lambda}_1 - a_0 \dddot{\lambda}_1 + \Delta t B (u_0 - u_{m0}). \]  

(3.40)

The third condition states that the derivative of \( \mathcal{J} \) with respect to \( \hat{F} \) should vanish. The condition leads to the following control equation:

\[ \frac{\partial \mathcal{J}}{\partial \hat{F}} = \dot{\lambda} = 0, \]  

(3.41)
which represents the discrete control equation and implies that $\frac{\partial \phi}{\partial \xi} = \frac{\partial \hat{\phi}}{\partial \xi}$ is the component of the vector $\hat{\lambda}$ corresponding to the global node numbering and the time step of $\xi$. In other words, if $\xi$ is $P_{kj}$, (3.41) becomes:

\[ \nabla (\xi = P_{kj}) \phi = \nabla (\xi = P_{kj}) L = \lambda (\gamma_k, t_j), \quad (3.42) \]

where $\lambda (\gamma_k, t_j)$ denotes the component of $\hat{\lambda}$ corresponding to the degree of freedom and the timestep of $P_{kj}$. Here, this works uses the solution of the adjoint problem for evaluating the gradient of $L$ with respect to $\xi$ in (3.42) in a semi-analytical manner.

### 3.4 Numerical Experiments

In this section, we investigate the performance of the presented inverse in a domain truncated by using a WABC in consideration of various factors. In the presented examples, we utilized the procedure shown in Algorithm 1 in Chapter 2, where the conjugate gradient in (2.78) and the Newton’s method in (2.85) are employed.

The domain size is set to be 200 m \times 60 m, and the mass density $\rho$ is uniform as 3000 kg/m$^3$. Two material profiles are considered in the presented numerical experiments. The material profile 1 is of a homogeneous background solid domain with two inclusions as in Fig. 3.1 with shear wave speeds of $V_{s1} = 353.55$ m/s, $V_{s2} = 800$ m/s, and $V_{s3} = 1000$ m/s. The material profile 2 is a 2-layered background solid with the same two inclusions, as shown in Fig. 3.2, with shear wave speeds of $V_{s1} = 353.55$ m/s, $V_{s2} = 223.6$ m/s, $V_{s3} = 800$ m/s, and $V_{s4} = 1000$ m/s.

The synthetic measured data $\hat{u}_m$ is computed by using an element size of 0.5 m for a targeted traction function. To avoid an inverse crime, we use an element size of 1 m for evaluating state solution $\hat{u}$ and the adjoint solution $\hat{\lambda}$ for estimated traction function. The time step of 0.001 s is used for computing $\hat{u}_m$, $\hat{u}$, and $\hat{\lambda}$.
First of all, four targeted dynamic tractions, $P_{1,2,3,4}(\gamma,t)$, are considered in the presented numerical experiments. They are tractions on the left and bottom boundaries of a homogeneous background domain, and they mimic incident seismic wavefield propagating from the left-bottom of the domain of an incident angle 45°. They vary over space and time, and their time signals at a specific location $\gamma = 160$ m are shown in Fig. 3.3(a-f). The time-dependent value of the targeted $P_{1,2,3}(\gamma,t)$ at any value of $\gamma$ on the boundaries, where the traction is applied, is a Ricker wavelet, where $P_1(\gamma,t)$ has the lowest central frequency (5 Hz), $P_2(\gamma,t)$ has a medium central frequency (10 Hz), and $P_3(\gamma,t)$ has the highest central frequency (20 Hz) among them. The total observation duration is 1.5 s in the inversion simulation of identifying targeted $P_{1,2,3}(\gamma,t)$. The time-dependent value of the targeted $P_4(\gamma,t)$ and its frequency content are shown in Fig. 3.3(g,h). This signal is a portion of a registered ground motions signal during the 1994 Northridge earthquake from the Pacific Earthquake Engineering Research Center (PEER) ground motion database [50]. The total observation duration is 6 s for the inversion simulation to identify the targeted $P_4(\gamma,t)$.

On the other hand, $P_5(\gamma,t)$ is a traction on the left and bottom boundaries of a two-layered background domain, and they mimic incident seismic wavefield propagating from the left-bottom of the domain of an incident angle 45° in the bottom layer. $P_5(\gamma = 160, t)$ has a central frequency of 5 Hz.

For the inversion process, we discretize estimated $P_{1,2,3}(\gamma,t)$ by using 391,500 control parameters (i.e., 261 (over space) $\times$ 1500 (over time)) and $P_4(\gamma,t)$ by using 1,566,000 control parameters.
Figure 3.3: (a,c,e,g) The time signal of $P_{1,2,3,4}(\gamma = 160,t)$, respectively; (b,d,f,h) the amplitude of Fourier Transform of $P_{1,2,3,4}(\gamma = 160,t)$, respectively.
(i.e., 261 (over space) × 6000 (over time)). The temporal and spatial intervals for the discretization are 0.001 s and 1 m, respectively, for all the estimated dynamic traction. The control parameters are set as zero at the initial time step.

In what follows, four examples of numerical investigations are presented.

- The first example is directed to compare the performance of inverting $P_1(\gamma,t)$ between Case 1—utilizing only the sensors on the top surface ($\Gamma_t$) of the domain—versus Case 2—using a borehole array of sensors on the right boundary ($\Gamma_r$) with the ones on the top surface $\Gamma_t$.

- The second example is focused on analyzing the performance of inverse modeling to reconstruct targeted dynamic traction with different central frequencies: Cases 2, 3, and 4 use $P_1(\gamma,t)$, $P_2(\gamma,t)$, and $P_3(\gamma,t)$, respectively, with their central frequencies of 5, 10, and 20 Hz.

- The third example investigates the inversion performance of Case 5 identifying $P_4(\gamma,t)$, of which $P_4(\gamma=160,t)$ is a realistic seismic signal. Namely, the performance in Case 5 will be compared with those in Case 2 to 4.

- Lastly, the fourth example examines the convergence of the estimated traction into the targeted $P_5(\gamma,t)$ by using the material profile 2 for the 2-layered background solid with two inclusions.

The summary of input parameters for every case presented in this chapter is shown in Table 3.1.

Table 3.1: Summary of all cases in Chapter 3.

<table>
<thead>
<tr>
<th>Case number</th>
<th>Material profile</th>
<th>Force profile</th>
<th>Vertical array</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>$P_1(\gamma,t)$</td>
<td>N</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>$P_1(\gamma,t)$</td>
<td>Y</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>$P_2(\gamma,t)$</td>
<td>Y</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>$P_3(\gamma,t)$</td>
<td>Y</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>$P_4(\gamma,t)$</td>
<td>Y</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>$P_5(\gamma,t)$</td>
<td>Y</td>
</tr>
</tbody>
</table>
For appraising the efficiency to reconstruct $P(\gamma, t)$ in the numerical results, the following error norm between estimated traction and its target is used:

$$\mathcal{E} = \frac{\int_0^T \int_\Omega |P(\gamma, t)_{\text{target}} - P(\gamma, t)_{\text{estimate}}|^2 d\Omega dt}{\int_0^T \int_\Omega |P(\gamma, t)_{\text{target}}|^2 d\Omega dt} \times 100\%.$$

\[ (3.43) \]

3.4.1 The exemplary forward wave responses

Fig. 3.4(a-d) show the wave responses in the whole computational domain induced by targeted $P_1(\gamma, t)$ in Case 1 at $t = 0.3$ s, 0.4 s, 0.5 s, and 0.6 s, respectively. Fig. 3.5 presents the wave responses caused by targeted $P_3(\gamma, t)$ in Case 6, in which material profile 2 is employed, at the same time steps.

The plots at $t = 0.3$ s in Fig. 3.4(a) and Fig. 3.5(a) point out that the dynamic input traction at the WABC mimics seismic incident, inclined plane waves. Also, Fig. 3.5(a) shows the change in the incident angle of the wave caused by the difference in the shear wave speeds in the two-layered medium. This behavior complies with the well-known Snell’s Law.

Fig. 3.4(b) and Fig. 3.5(b) display the waves passing through the first inclusion, which is 60 m from the left boundary. Fig. 3.4(c) and Fig. 3.5(c) show the wave motions passing through the second inclusion, which is 120 m from the left boundary. Finally, Fig. 3.4(d) and Fig. 3.5(d) exhibit the waves arriving at the right boundary.
Figure 3.4: (a-d) The wave motions of $u(x, y, t) [m]$, at $t = 0.3$ s, $0.4$ s, $0.5$ s, and $0.6$ s, respectively, in Case 1.
Figure 3.5: (a-d) The wave motions of $u(x, y, t) \text{[m]}$, at $t = 0.3 \text{ s}$, $0.4 \text{ s}$, $0.5 \text{ s}$, and $0.6 \text{ s}$, respectively, in Case 6.
3.4.2 Demonstration of the relation between the traction and $u^1$ on a WABC boundary.

Fig. 3.6 compares $P_1(\gamma, t)$ with $\frac{2G}{V} \frac{\partial u}{\partial t}$—per (3.6)—on $\Gamma_l$, $\Gamma_b$, and $\Gamma_r$ in an entirely-homogeneous domain. As shown in Fig. 3.6(b), $\frac{2G}{V} \frac{\partial u}{\partial t}$ (except for those of waves reflected from $\Gamma_l$ arriving at $\Gamma_l$ and $\Gamma_b$) is in an excellent agreement with the traction $P_1(\gamma, t)$. Therefore, our underlying rationale of (3.6) is justified.

![Figure 3.6: (a) Input traction $P_1(\gamma, t)$ and (b) $\frac{2G}{V} \frac{\partial u}{\partial t}$, in N/m².](image)

3.4.3 The verification of the inverse modeling

The theoretical derivation of the adjoin and control equations and the numerical implementation are verified prior to the parametric studies. The gradient obtained by our semi-analytical DTO approach is compared with that from the finite difference (FD) procedure, which uses the following equation:

$$\frac{\partial L}{\partial \xi}_{FD} = \frac{L(\xi + h) - L(\xi)}{h}, \quad (3.44)$$

where $\xi$ indicates a control parameter. In this verification, we used the target $P_1(\gamma, t)$, material profile 1, and only one sensor in the center of $\Gamma_t$. Fig. 3.7 reveals an outstanding agreement
between the normalized gradient (i.e., $\nabla_\xi L / |\nabla_\xi L|$) that is calculated by using the presented DTO approach and that by the FD approach. Both of them are computed at the first inversion iteration. Consequently, the theoretical derivation and mathematical implementation are trustable.

Figure 3.7: Comparison between the gradients generated by the presented inverse modeling and the FD approximation

### 3.4.4 Example 1: Investigating the inversion performance with/without a vertical array of sensors

In this example, we compare the performance of the presented inverse modeling of identifying $P_1(\gamma, t)$ (i) when we use the sensors only on the top surface $\Gamma_t$ (Case 1) versus (ii) when we add a vertical (e.g., at borehole) array of sensors on the boundary on the right side $\Gamma_r$ (Case 2). We used the material profile 1 and 50 sensors on $\Gamma_t$ in both Cases. In Case 2, 15 sensors are added on $\Gamma_r$.

Fig. 3.8 shows the targeted $P_1(\gamma, t)$ and its reconstructed counterpart obtained by using sensors on only one boundary ($\Gamma_t$) versus two boundaries ($\Gamma_t$ and $\Gamma_r$), in Case 1 and 2, respectively. When Case 1 is tested, as shown in Fig. 3.8(b), the incident, inclined plane wave passing the bottom
boundary $\Gamma_b$ cannot be fully recovered. However, Fig. 3.8(c) shows that by adding the vertical array of sensors on $\Gamma_r$, the incident traction is accurately reconstructed.

![Graph showing traction reconstruction](image)

Figure 3.8: Example 1: (a) Target and (b,c) Reconstructed $P_1(\gamma, t)$ in N/m², for Case 1 and 2 at the 1000th iteration. Horizontal and vertical axis in the contour plot represent, respectively, the numbering of the discretized points over space ($\gamma$) and time ($t$) of the distribution of $P_1(\gamma, t)$.

Fig. 3.9 shows that, by using the vertical array of sensors in Case 2, the terminal value of $\mathcal{E}$ of the inversion becomes much smaller than that in Case 1. After 1000 iterations, the final value of $\mathcal{E}$ is $9.39\%$ in Case 1, and that in Case 2 is $1.72\%$, which shows a reduction of $7.67\%$ in $\mathcal{E}$. Thus, we suggest that using the vertical array of sensors improves the inversion performance in the presented problem setting.
Figure 3.9: Example 1 - After 1000 iterations, the final value of $\varepsilon$ is 9.39% in Case 1, and that in Case 2 is 1.72%, which shows a reduction of 7.67% in $\varepsilon$. 
3.4.5 Example 2: Investigating the inversion performance with respect to the dominant frequency of $P(\gamma,t)$

This example studies the accuracy of the inversion with respect to the dominant frequency of $P(\gamma,t)$. We used 50 sensors on the top surface $\Gamma_t$, 15 sensors on the vertical array along $\Gamma_r$, and material profile 1. In addition to the Case 2 in Example 1, where the central frequency of the dynamic traction is 5 Hz, we examined Case 3 and 4, which have a central frequency of 10 and 20 Hz, respectively.

Fig. 3.10(a,c,e) show the targeted dynamic traction for Cases 2, 3, and 4, respectively, while Fig. 3.10(b,d,f) show their reconstructed counterparts in each case. Fig. 3.11 reveals that, the higher frequency content the traction has, the higher the terminal value of $\mathcal{E}$ is obtained. Accordingly, after 2000 iterations, the terminal value of $\mathcal{E}$, 1.32%, for reconstructing $P_1(\gamma,t)$ in Case 2 is smaller than its counterparts, 3.38%, of rebuilding $P_2(\gamma,t)$ in Case 3 and 11.02% for reconstructing $P_3(\gamma,t)$ in Case 4. Thus, the inversion modeling presented in this dissertation suffers more from solution multiplicity when it identifies traction of a higher dominant frequency, as discussed in the previous chapter earlier.

Fig. 3.12, Fig. 3.13, and Fig. 3.14 show the snapshots of the wave responses in the entire computational domain induced by the targeted $P_{1,2,3}(\gamma,t)$ and their reconstructed counterparts, respectively. All figures present excellent agreements between wave responses induced by the targeted and reconstructed traction even in Case 4, of which terminal value of $\mathcal{E}$ is the largest among the Cases 2-4.

3.4.6 Example 3: Examining the feasibility of the presented inverse modeling to reconstruct a realistic seismic signal $P_4(\gamma,t)$

In this example, we study the performance of identifying targeted $P_4(\gamma,t)$, which uses a realistic seismic signal (Case 5). In Case 5, we use the material profile 1 with 50 sensors on $\Gamma_t$ and 15 sensors on $\Gamma_r$. 
Figure 3.10: Example 2: (a) Target and (b) Reconstructed $P_1(\gamma, t)$ in N/m$^2$; (c) Target and (d) Reconstructed $P_2(\gamma, t)$ in N/m$^2$; and (e) Target and (f) Reconstructed $P_3(\gamma, t)$ in N/m$^2$, for Cases 2-4 at the 1000th iteration.
Figure 3.11: Example 2 - After 2000 iterations, the terminal value of $\mathcal{E}$, 1.32%, for reconstructing $P_1(\gamma,t)$ in Case 2 is smaller than its counterparts, 3.38%, of rebuilding $P_2(\gamma,t)$ in Case 3 and 11.02% for reconstructing $P_3(\gamma,t)$ in Case 4.

Fig. 3.15 shows the excellent agreement between the targeted and reconstructed dynamic tractions, $P_4(\gamma,t)$, in Case 5. The inversion performance in Case 5 is compared with that in Case 2. The Cases 2 and 5 use the same material and sensor configurations, but different dynamic traction $P_1(\gamma,t)$ and $P_4(\gamma,t)$, respectively. Fig. 3.16 shows the values of $\mathcal{L}$ and $\mathcal{E}$ decreasing over iterations. After 1000 iterations, as shown in Fig. 3.16(b), the terminal value of $\mathcal{E}$ in Case 5 is 0.38%, which is smaller than that in Case 2 (1.72%).

We note that $P_4(\gamma,t)$ has a lower dominant frequency than $P_1(\gamma,t)$. Consequently, the numerical experiments shows again that our minimizer suffers less from solution multiplicity when it has to reconstruct a target traction with a lower dominant frequency. Namely, even though $P_4(\gamma,t)$ is a more complex seismic signal, $P_4(\gamma,t)$ is identified more accurately than $P_1(\gamma,t)$ because $P_4(\gamma,t)$ has a lower dominant frequency than $P_1(\gamma,t)$. 
Figure 3.12: Example 2- Case 2: Wave responses, $u(x,y,t)[m]$, in the domain induced by (left) the targeted $P_1(\gamma,t)$ and (middle) its reconstructed counterpart, and (right) the difference between them at (a-c) 0.3 s, (d-f) 0.4 s, (g-i) 0.5 s, and (j-l) 0.6 s.
Figure 3.13: Example 2 - Case 3: Wave responses, $u(x,y,t)$ [m], in the domain induced by (left) the targeted $P_2(\gamma,t)$ and (middle) its reconstructed counterpart, and (right) the difference between them at (a-c) 0.3 s, (d-f) 0.4 s, (g-i) 0.5 s, and (j-l) 0.6 s.
Figure 3.14: Example 2 - Case 4: Wave responses, \( u(x,y,t)[m] \), in the domain induced by (left) the targeted \( P_3(\gamma,t) \) and (middle) its reconstructed counterpart, and (right) the difference between them at (a-c) 0.3 s, (d-f) 0.4 s, (g-i) 0.5 s, and (j-l) 0.6 s.
Fig. 3.15: Example 3: (a) Target and (b) Reconstructed $P_4(\gamma, t)$, in N/m$^2$, for Case 5 at the 1000th iteration.

Fig. 3.17 and Fig. 3.18 show that the wave responses induced by the targeted $P_4(\gamma, t)$ are in excellent agreement with those induced by its reconstructed counterpart in Case 5.

3.4.7 Example 4: Investigating the inversion performance in a 2-layered background domain

This example focuses on examining the performance of inverting $P_5(\gamma, t)$ in a more complex background domain than the single-layered, homogeneous background domain. To this end, we used the material profile 2, a vertical array of 15 sensors on $\Gamma_r$, and 50 sensors on $\Gamma_t$ in Case 6. Case 6 is compared to Case 2, in which the material profile 1 is used with the traction $P_1(\gamma, t)$ of the same dominant frequency as $P_5(\gamma, t)$ and the same sensor configuration as Case 6.

Fig. 3.19 shows the excellent agreement between the targeted $P_5(\gamma, t)$ and its reconstructed dynamic traction. Fig. 3.20 shows the value of $\mathcal{E}$ over iterations in Case 2 and 6 for material profiles 1 and 2, respectively. In Case 6, the traction mimics the incident plane wave, and even
Figure 3.16: Example 3 - $\mathcal{L}$ and $\mathcal{E}$ comparing Case 2 and Case 5.
Figure 3.17: Example 3: Wave responses, \( u(x, y, t)[\text{m}] \), in the domain induced by (left) the targeted \( P_4(\gamma, t) \) and (right) its reconstructed counterpart at (a-b) 1.0 s, (c-d) 2.5 s, (e-f) 3.0 s, and (g-h) 3.5 s.
Figure 3.18: Example 3: Wave responses, \( u(x, y, t)[m] \), in the domain induced by (left) the targeted \( P_4(\gamma, t) \) and (right) its reconstructed counterpart at (a-b) 3.75 s, (c-d) 4.5 s, (e-f) 5.25 s, and (g-h) 6.0 s.
with a more complex domain, the final value of $E$ is 1.91%. The difference between the terminal values of $E$ in Cases 2 and 6 is only less than 0.20%.

Fig. 3.21 shows that the wave responses of $u_m$ due to the targeted $P_5(\gamma,t)$ are in excellent agreement with those of $u$ due to the reconstructed one at sensors on the top surface and right boundary, where the vertical sensor array is localized.

Fig. 3.22 presents the ground responses, $u(x,y,t)$, in the domain induced by the targeted $P_5(\gamma,t)$ and its reconstructed counterpart for Case 6 at $t = 0.3$ s, 0.4 s, 0.5 s, and 0.6 s. In general, the responses are in an excellent agreement with each other.

![Figure 3.19: Example 4: (a) Target and (b) Reconstructed $P_5(\gamma,t)$, in N/m², for Case 6 at the 1000th iteration.](image)
Figure 3.20: Example 4 - The final value of $\varepsilon$, in Case 6, is 1.91%, and that in Case 2 is 1.72%.

Figure 3.21: $u_m$ and $u$ at sensors (a) on the top surface measured at $x = 99$ m and (b) on the right boundary measure at $\gamma = 288$ m for Case 6.
3.5 Summary

This chapter presents a robust mathematical and computational modeling that identifies complex, incoherent seismic incident wavefield in a 2D solid, truncated by a WABC. The DTO approach is utilized to solve the adjoint problem in the inverse modeling procedure.

Numerical results depict the performance of this novel inversion approach in a domain enclosed by WABC. First, targeted traction—mimicking incident, inclined plane waves—cannot be
fully reconstructed by using only the sensors on the top surface because wave motions induced by the dynamic traction at the bottom boundary do not reach the top surface. Second, the presented numerical solver suffers from solutions multiplicity more when it identifies dynamic traction of higher frequency content. Third, even though the dynamic traction is of a more complex real earthquake signal, our inversion solver shows an excellent inversion performance because the signal has a low-frequency content. Lastly, when a more complex material profile is used, the inversion solver can still reconstruct targeted dynamic traction.
CHAPTER 4
SIMPLE JOINT INVERSION USING GENETIC ALGORITHM (GA)

4.1 Problem Definition
This study is aimed at identifying both the spatial distribution of a material property of a Timoshenko continuous beam-based bridge model and the profile of a moving vibrational source by using the GA-based inverse modeling. This work considers a one-dimensional Timoshenko beam supported at four locations by a hinge and rollers (see Fig. 4.1).

![Diagram of beam subject to a moving vibrational load and its vertical deflection and slope.](image)

(a) A beam subject to a moving vibrational load

(b) The vertical deflection $w(x, t)$ and slope $\psi(x, t)$ of a beam

Figure 4.1: Problem configuration of Chapter 4.

Its governing wave equations are the followings (for brevity, the temporal and spatial dependencies of variables are omitted in the following equations):
\[ \frac{\partial}{\partial x} \left\{ GAK_s \left( \frac{\partial w}{\partial x} - \psi \right) \right\} - \rho A \frac{\partial^2 w}{\partial t^2} = -q, \quad (4.1) \]
\[ GAK_s \left( \frac{\partial w}{\partial x} - \psi \right) + \frac{\partial}{\partial x} \left( EI \frac{\partial \psi}{\partial x} \right) - \rho I \frac{\partial^2 \psi}{\partial t^2} = 0, \quad (4.2) \]

where \( x \in (0, L) \) denotes a position in the beam (\( L \) is the total length of the beam); \( t \in (0, T) \) denotes time (\( T \) is the total observation time); \( w(x, t) \) is the total deflection of a beam at \( x \) and \( t \); \( \psi(x, t) \) is the slope of a beam caused by bending only; \( E(x) \) is Young’s modulus; \( G(x) \) is the shear modulus; \( \rho(x) \) is the mass density; and \( A(x) \) and \( I(x) \) denote the cross-sectional area and the second moment of inertia, respectively; \( K_s(x) \) denotes the Timoshenko shear factor; and \( q(x, t) \) is the excitation force applied from a wave source (e.g., a moving vehicle) on the beam.

The beam is supported at multiple locations by a hinge and rollers, and, hence, the boundary conditions (BCs) of the beam are:

\[ w(a, t) = 0, \quad 0 \leq t \leq T, \quad (4.3) \]
\[ EI \frac{\partial \psi}{\partial x}(a, t) = 0, \quad 0 \leq t \leq T, \quad (4.4) \]

where \( a \) denotes the location of either a hinge or a roller. Equations (4.3) and (4.4) indicate that the deflection and the bending moment of the beam vanish at the locations of the hinge and roller supports. The beam is initially at rest: the initial-value conditions are:

\[ w(x, 0) = 0, \quad \frac{\partial w}{\partial t}(x, 0) = 0, \quad (4.5) \]
\[ \psi(x, 0) = 0, \quad \frac{\partial \psi}{\partial t}(x, 0) = 0. \quad (4.6) \]

This paper considers that there is a moving vibrational force exerted on the beam. Namely, the vibrational force in (4.1) is defined as:

\[ q(x, t) = F(t)H(x, t), \quad (4.7) \]
where the time-harmonic excitation of the force is defined as:

\[ F(t) = P \sin(2\pi ft), \]

where \( P \) is the amplitude of the sinusoidal temporal variation of the force; \( f \) is its frequency. In (4.7), the time-dependent (i.e., moving) spatial variation term of \( q(x,t) \) is defined as:

\[ H(x,t) = (\cos(z) + 1)e^{-|2\times z^2|}, \quad z = x - x_0 - \omega t, \]

where \( x_0 \) denotes the initial position of the centroid of \( H(x,t) \) at the initial observation time \( (t = 0) \); and \( \omega \) is its moving speed. An example of how \( H(x,t) \) changes over time and space is shown in Fig. 4.2.

![Figure 4.2](image-url)

Figure 4.2: Two snapshots of \( H(x,t) \) at \( t = 0 \) and \( 1 \) s using \( x_0 = 50 \) m and \( \omega = 20 \) m/s.

In this study, the values of the amplitude \( P \) and frequency \( f \) of \( F(t) \) are unknown and set to be reconstructed while the initial position \( x_0 \) and moving speed \( \omega \) of \( H(x,t) \) are known in advance before the presented inversion solver is executed. Here, we assume that, in practice, engineers can easily estimate \( x_0 \) and \( \omega \) by using visual footage made by traffic-surveillance cameras and transfer
the known information of $x_0$ and $\vartheta$ to the inversion solver. In contrast, the amplitude $P$ and $f$ are hard to estimate from the video footage because $P$ is affected by the total weight of a vehicle, including its passengers and freights, and $f$ is associated with the vehicle’s internal vibration. Thus, this work identifies $P$ and $f$ of a wave source as well as the material property of a beam.

4.2 Forward Wave Modeling

This section presents the finite element modeling for obtaining the numerical solutions of the governing wave equations.

4.2.1 Finite Element Method

The governing wave equations (4.1) and (4.2) are multiplied by test functions $u(x)$ and $v(x)$, respectively, and integrated over the domain $(0, L)$. Then, they result in the following weak forms:

\[
- \int_0^L \left( GAK_s \frac{\partial w}{\partial x} \frac{\partial u}{\partial x} \right) \, dx + \int_0^L \left( GAK_s \varphi \frac{\partial u}{\partial x} \right) \, dx \\
- \int_0^L \left( \rho Au \frac{\partial^2 w}{\partial t^2} \right) \, dx = - \int_0^L (uq) \, dx,
\]

\[
\int_0^L \left( GAK_s v \frac{\partial w}{\partial x} - GAK_s v \varphi \right) \, dx - \int_0^L \left( EI \frac{\partial v}{\partial x} \frac{\partial \varphi}{\partial x} \right) \, dx \\
- \int_0^L \left( \rho Iv \frac{\partial^2 \varphi}{\partial t^2} \right) \, dx = 0.
\]

Next, the test and trial functions are approximated as follows:

---

1Please also note that engineers can choose to use the data made during a particular observation time (e.g., midnight), when there is only a single vehicle on an inspected bridge, for the presented inversion solver. From the vibrational data that are obtained during such an observational time slot, we invert for the control parameters of only one wave source and the material properties of an inspected structure. Joint inversion is more plausible in a case with only one unknown wave source than a case with multiple unknown sources because, the more control parameters are to be inverted for, an inverse problem is more likely to suffer from the solution multiplicity.
\[ u(x) \simeq u^T \phi(x), \quad v(x) \simeq v^T g(x), \]
\[ w(x,t) \simeq \phi(x)^T w(t), \quad \psi(x,t) \simeq g(x)^T \Psi(t), \]
\[ (4.12) \]
where \( w(t) \) and \( \Psi(t) \) are the vectors of unknown nodal deflections and slopes of a beam, respectively, and \( \phi(x) \) and \( g(x) \) are the vectors of global basis functions that are made of shape functions in the local coordinate of each element. In this work, we use the standard Lagrange 4-noded cubic shape functions to approximate \( w(x,t) \) and \( u(x) \), and the Lagrange 3-noded quadratic shape functions to approximate \( \psi(x,t) \) and \( v(x) \) [52].

By virtue of the approximation of the test and trial functions as shown in (4.12), the weak form (4.10) and (4.11) reduce to the following semi-discrete system:

\[ \mathbf{M} \ddot{\mathbf{d}}(t) + \mathbf{K}\mathbf{d}(t) = \mathbf{Q}(t), \]
\[ (4.13) \]
where \( (\cdot) \) denotes the second-order derivative of a subtended variable with respect to \( t \); \( \mathbf{M} \) denotes a global mass matrix; \( \mathbf{K} \) denotes a global stiffness matrix; \( \mathbf{Q} \) denotes a global load vector; and \( \mathbf{d}(t) \) is the solution vector composed by \( w(t) \) and \( \Psi(t) \). In (4.13), the vectors and matrices are defined as:

\[ \mathbf{d}(t) = \begin{bmatrix} w(t) \\ \Psi(t) \end{bmatrix}, \quad \mathbf{Q}(t) = \begin{bmatrix} \int_0^{L} \phi q \, dx \\ 0 \end{bmatrix}, \]
\[ (4.14) \]
\[ \mathbf{M} = \begin{bmatrix} \int_0^{L} \rho A \phi \phi^T \, dx & 0 \\ 0 & \int_0^{L} \rho I g g^T \, dx \end{bmatrix}, \]
\[ (4.15) \]
\[ \mathbf{K} = \begin{bmatrix} \int_0^{L} G A K_s \phi' \phi'^T \, dx & \int_0^{L} G A K_s \phi' g^T \, dx \\ \int_0^{L} G A K_s g \phi'^T \, dx & \int_0^{L} (G A K_s g g^T + E I g' g'^T) \, dx \end{bmatrix}, \]
\[ (4.16) \]
where \( (\cdot) \) denotes the derivative of a subtended variable with respect to \( x \).
We solve the time-dependent ordinary differential equation (4.13) in every \(i\)-th discrete time step as:

\[
\mathbf{M}\ddot{\mathbf{d}}_i + \mathbf{K}\mathbf{d}_i = \mathbf{Q}_i. \tag{4.17}
\]

By applying the initial-value conditions (4.5) and (4.6) onto (4.17), the solution vector at the initial time step is obtained by solving the following:

\[
\mathbf{M}\ddot{\mathbf{d}}_1 = \mathbf{Q}_1. \tag{4.18}
\]

After the initial time step, this work solves the system of equation for each time step using Newmark implicit time integration (i.e., the average acceleration scheme), which results in the unconditionally-stable numerical solution of wave responses [53]. The solution vector of the \(i\)-th time step is related to its previous time step as:

\[
\mathbf{d}_i = \mathbf{d}_{i-1} + \frac{1}{2}[0.5\dot{\mathbf{d}}_{i-1} + 0.5\ddot{\mathbf{d}}_i](\Delta t)^2, \tag{4.19}
\]

and

\[
\dot{\mathbf{d}}_i = \dot{\mathbf{d}}_{i-1} + [0.5\ddot{\mathbf{d}}_{i-1} + 0.5\dddot{\mathbf{d}}_i](\Delta t), \tag{4.20}
\]

where \(\Delta t\) denotes the size of a time step. By plugging (4.19) into (4.17), it turns into the following:

\[
[M + 0.25K(\Delta t)^2]\dddot{\mathbf{d}}_i = \mathbf{Q}_i - K[\mathbf{d}_{i-1} + \dot{\mathbf{d}}_{i-1}(\Delta t) + 0.25\ddot{\mathbf{d}}_{i-1}(\Delta t)^2]. \tag{4.21}
\]

By using (4.21), this work solves for \(\dddot{\mathbf{d}}_i\). Then, the values of \(\mathbf{d}_i\) and \(\dot{\mathbf{d}}_i\) can be updated by using, respectively, (4.19) and (4.20).

**4.2.2 Verification of the forward wave modeling**

Prior to our investigation on the performance of the presented joint inversion, we verify our FEM wave solver, written in MATLAB, by comparing our solution with the reference solution
calculated by another wave solver, written in Fortran, used for a previous work by Karve et al. [52]. This verification considers a 10 m-long Timoshenko beam, which is simply supported by a hinge and a roller. The beam is discretized by using 100 elements (each element is 0.1 m long), and the total observation duration $T$ is 1.0 s with a time step $\Delta t$ of 0.001 s. The beam is homogeneous and has the following properties: Young’s modulus ($E$) of $2.5 \times 10^{10}$ Pa, shear modulus ($G$) of $1 \times 10^{10}$ Pa, mass density ($\rho$) of 2500 kg/m$^3$, cross-section area ($A$) of 0.1 m$^2$, second moment of inertia ($I$) of 0.0013 m$^4$, and shear factor ($K_s$) of 0.8333. In this verification, a uniformly-distributed excitational loading is exerted on all the elements of the beam, and it is defined as $q(x,t) = 100\sin(2\pi ft)$ N/m with its frequency $f$ of 2 Hz. Fig. 4.3 shows an excellent agreement between the displacement field of the wave response, in the center of the beam, from our FEM wave solver and that from the reference code. Hence, the forward wave modeling presented in this work is reliable and can be used in the presented inversion modeling.

Figure 4.3: Comparison between $u(5,t)$ generated by our FEM wave solver and that by a reference code for a simply-supported beam of its length 10 m subject to a uniformly-distributed sinusoidal loading of its frequency 2 Hz.
4.3 Inverse Modeling

The objective of the inverse modeling in this work is to estimate the values of the control parameters—characterizing a moving wave source and the spatial distribution of a material property of a beam—that minimize the following misfit functional:

\[ L = \int_0^T \sum_{j=1}^{NS} |u_{m}^j(t) - u_j(t)|^2 \, dt. \]  

(4.22)

In (4.22), \( T \) is the total observation time; \( NS \) is the number of sensors; \( u_{m}^j \) and \( u_j \) are, respectively, the measured response, due to targeted control parameters, and the computed response, due to estimated parameters, at the location of the \( j \)-th sensor and time \( t \). In this computational study, we synthetically create the measured response data, \( u_{m}^j(t) \), by using our FEM solver with targeted control parameters.

In this work, the genetic algorithm (GA) is employed to estimate unknown values of control parameters that correspond to the minimal value of the misfit functional (4.22). We choose to use the GA because of its effectiveness for an optimization or inverse problem with respect to a small number of control variables—in our presented numerical experiments, the number of control parameters is sufficiently small (at most 11).

The GA involves a series of generations (i.e., inversion iterations). At each generation, the GA explores the profiles of a given number of individuals, each of which contains a set of all the control parameters in an optimization or inverse problem. In the presented inverse modeling, each individual consists of following control parameters—the amplitude and frequency of a moving wave source and the Young’s modulus of each segment in a Timoshenko beam model, which is assumed to be piece wisely homogeneous.

The total number of generations is referred to as GN in this paper. In the beginning of the GA, it is assigned the value of GN and a given number of individuals, which is referred to as the population size (PS). As mutation and cross-over among the individuals diversify each population, the GA
explores the fittest individual. At the last generation, the GA returns the fittest individual that leads to the smallest value of the misfit functional, and its control parameters will be the final inversion solution. This work uses the built-in GA function in MATLAB, and it autonomously conducts the mutation and cross-over of individuals, for each of which $u_j(t)$ at each $j$-th measurement location is computed by using our forward wave solver.

4.4 Numerical Experiments

This section shows numerical examples, investigating the performance of the presented GA-based joint inversion method with respect to various factors. The first example is focused on the performance of the presented inversion solver for estimating the values of five control parameters—two for a moving source and three for the elastic moduli of the beam structure’s three segments. The second example examines the inversion solver for estimating the values of eleven control parameters—two for a moving source and nine for the elastic moduli of the beam structure’s nine segments. In these two examples, this work investigates the effects of the population size (PS) and/or the number of sensors (NS) on the performance of the presented joint inversion algorithm.

In all the examples, we consider a Timoshenko beam bridge model, of which the total extent is 100 m. It is supported by a hinge at $x$ of 0 m, and three rollers at $x$ of 33.3, 66.7, and 100 m, respectively. It is discretized by using 180 elements (an element size is about 0.56 m, and each element contains a set of 4 nodes for approximating $w$ and another set of 3 nodes for approximating $\Psi$), and the time step of 0.001 s is used in the FEM solver. The total observation duration $T$ is 1.0 s, and the sensors are sparsely distributed along the beam with uniform spacing.

In the presented inversion simulations, it is assumed that the inversion solver uses the following $a$-priori known, uniformly-distributed material properties of a beam—$\rho$ of 2500 kg/m$^3$, $A$ of 0.1 m$^2$, $I$ of 0.0013 m$^4$, and $K_s$ of 0.8333. On the other hand, the value of $E$ is unknown, and it could vary with respect to the location of the beam model. Thus, its spatial distribution is to be identified
during the inverse modeling. In the presented numerical examples of the joint inversion, a targeted moving wave source is known to move with its moving speed $\vartheta$ of 20 m/s toward the right-hand side of the beam from its initial position at $x_0$ of 50 m. In contrast, its amplitude ($P$) of 100 N/m and frequency ($f$) of 20 Hz are unknown, and their values will be reconstructed during the presented GA process while the upper and lower limits of $P$ and $f$ are set by using $\pm$ 50% deviations of their targeted values. That is, their values are bounded as $50 \leq P \leq 150$ N/m and $10 \leq f \leq 30$ Hz during the presented inversion process.

As the postprocessing of the inversion results, the error between each target control parameter and its corresponding estimated solution of the fittest individual at each generation is computed as:

$$\varepsilon = \frac{|A \text{ targeted value} - A \text{ estimated value from the GA}|}{|A \text{ targeted value}|} \times 100\%.$$  

(4.23)

An averaged error norm for all the control parameters of the fittest individual at each generation is also defined as:

$$\bar{\varepsilon} = \frac{\sum_{k=1}^{NP} \varepsilon_k}{NP} \%.$$  

(4.24)

where $NP$ denotes the total number of target control parameters of an individual, and $k$ denotes the $k$-th control parameter of an individual, and $\varepsilon_k$ is the error, defined in (4.23), of the inversion of the $k$-th control parameter.

### 4.4.1 Example 1 (Cases 1 to 5): joint inversion of two source parameters and three stiffness parameters in a bridge comprised of three piece wisely-homogeneous segments

In this example, we consider that a continuous beam model consists of three piece wisely-homogeneous segments (see Fig. 4.4), and each segment’s Young’s modulus ($E$) is estimated by the presented joint inversion method. The targeted $E$ of the beam’s first segment ($0 \leq x \leq 33.3$ m) is $1.8 \times 10^{10}$ Pa, and it is smaller than those ($2.5 \times 10^{10}$ Pa) of the other segments. This reduced stiffness in the first segment represents a structural anomaly, e.g., corrosion-induced reduced stiffness.
Fig. 4.5 shows the snapshots of a targeted moving source function $q(x,t)$ and its corresponding wave responses of a displacement field in the entire beam at 0.28 and 0.82 seconds, considering the targeted parameters of the source and the material of this example. Fig. 4.6 shows the frequency contents of wave responses (up to $t$ of 0.3 s) measured at sensors that are located, respectively, in front of ($x = 60$ m) and behind ($x = 40$ m) the targeted moving source whose excitational frequency is 20 Hz. The wave response, measured at $x$ of 60 m, shows the forward frequency shift (20.31 Hz) of its dominant frequency, and the other, at $x$ of 40 m, shows the backward frequency shift (19.03 Hz). The frequency shifts are attributed to the Doppler effect of the wave responses induced by a moving source [54].

The value of the estimated $E$ in each segment of the beam is bounded as $1.7 \times 10^{10}$ Pa $\leq E \leq 2.6 \times 10^{10}$ Pa during the GA-based inversion simulation. Please note that $E$ is quite unlikely to exceed its designed value of $2.5 \times 10^{10}$ Pa during the lifespan of a bridge whereas it could become smaller than its designed value due to structural damage.

In this Example 1, Cases 1 to 5 are examined to detect the targeted control parameters by using five different combinations of PS and NS, and we used GN of 50 for all Cases 1 to 5. Both Cases 1 and 2 use NS of 45 (the spacing between neighboring sensors is about 2.2 m), but each of them uses PS of 100 and 50, respectively. On the other hand, Cases 3 to 5 use PS of 50, but each of them uses NS of 30, 15, and 10, respectively (the sensor spacings are 3.3, 6.3, and 9.2 m, respectively, in Cases 3 to 5). Table 4.1 shows the reconstructed values of the control parameters of the best-fit individual at the final generation in Cases 1 to 5. The table also shows the error, defined in (4.23),
Figure 4.5: Snapshots of the exemplary wave response, $u(x,t)$, and the targeted source function, $q(x,t)$, in Example 1 considering the targeted material profile.
Table 4.1: Example 1: joint inversion of two source parameters and three stiffness parameters in a bridge comprised of three piece wisely-homogeneous segments (GN = 50 for all the cases 1 to 5).

<table>
<thead>
<tr>
<th>Cases</th>
<th>Parameters</th>
<th>Value</th>
<th>Error, ε</th>
<th>Average Error, ε</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target</td>
<td>P (N/m)</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>f (Hz)</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_1$ (Pa)</td>
<td>$1.8 \times 10^{10}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_2$ (Pa)</td>
<td>$2.5 \times 10^{10}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$E_3$ (Pa)</td>
<td>$2.5 \times 10^{10}$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case 1</td>
<td>P (N/m)</td>
<td>100.07</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>f (Hz)</td>
<td>20.00</td>
<td>0.0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$PS = 100$</td>
<td>$E_1$ (Pa)</td>
<td>$1.80 \times 10^{10}$</td>
<td>0.2%</td>
</tr>
<tr>
<td></td>
<td>$NS = 45$</td>
<td>$E_2$ (Pa)</td>
<td>$2.50 \times 10^{10}$</td>
<td>0.0%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_3$ (Pa)</td>
<td>$2.51 \times 10^{10}$</td>
<td>0.3%</td>
</tr>
<tr>
<td>Case 2</td>
<td>P (N/m)</td>
<td>100.01</td>
<td>0.0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>f (Hz)</td>
<td>20.01</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$PS = 50$</td>
<td>$E_1$ (Pa)</td>
<td>$1.79 \times 10^{10}$</td>
<td>0.4%</td>
</tr>
<tr>
<td></td>
<td>$NS = 45$</td>
<td>$E_2$ (Pa)</td>
<td>$2.51 \times 10^{10}$</td>
<td>0.5%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_3$ (Pa)</td>
<td>$2.49 \times 10^{10}$</td>
<td>0.5%</td>
</tr>
<tr>
<td>Case 3</td>
<td>P (N/m)</td>
<td>100.19</td>
<td>0.2%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>f (Hz)</td>
<td>20.01</td>
<td>0.0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$PS = 50$</td>
<td>$E_1$ (Pa)</td>
<td>$1.80 \times 10^{10}$</td>
<td>0.1%</td>
</tr>
<tr>
<td></td>
<td>$NS = 30$</td>
<td>$E_2$ (Pa)</td>
<td>$2.52 \times 10^{10}$</td>
<td>0.6%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_3$ (Pa)</td>
<td>$2.48 \times 10^{10}$</td>
<td>0.9%</td>
</tr>
<tr>
<td>Case 4</td>
<td>P (N/m)</td>
<td>98.81</td>
<td>1.2%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>f (Hz)</td>
<td>20.00</td>
<td>0.0%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$PS = 50$</td>
<td>$E_1$ (Pa)</td>
<td>$1.80 \times 10^{10}$</td>
<td>0.1%</td>
</tr>
<tr>
<td></td>
<td>$NS = 15$</td>
<td>$E_2$ (Pa)</td>
<td>$2.51 \times 10^{10}$</td>
<td>0.4%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_3$ (Pa)</td>
<td>$2.49 \times 10^{10}$</td>
<td>0.3%</td>
</tr>
<tr>
<td>Case 5</td>
<td>P (N/m)</td>
<td>100.84</td>
<td>0.8%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>f (Hz)</td>
<td>20.01</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td></td>
<td>$PS = 50$</td>
<td>$E_1$ (Pa)</td>
<td>$1.79 \times 10^{10}$</td>
<td>0.4%</td>
</tr>
<tr>
<td></td>
<td>$NS = 10$</td>
<td>$E_2$ (Pa)</td>
<td>$2.53 \times 10^{10}$</td>
<td>1.1%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$E_3$ (Pa)</td>
<td>$2.47 \times 10^{10}$</td>
<td>1.0%</td>
</tr>
</tbody>
</table>
between the reconstructed value of each parameter and its targeted value and the average error, defined in (4.24), of all the parameters. Cases 2 to 5 demonstrate the performance of the inversion solver with respect to the number of sensors (or NS). Namely, as shown in Fig. 4.7, the average error for the best-fit individual at the final generation tends to increase as NS decreases. Table 4.1 shows that Case 1 results in the smallest average error because Case 1 uses larger values of PS and NS than the other cases.

In what follows, we describe the inversion performance of Case 1, which shows the best performance among Cases 1 to 5. Fig. 4.8 shows how the value of the misfit functional for the best-fit individual changes over the GA iterations in Case 1. Figs. 4.9 and 4.10 show the histograms of estimated control parameters of all the individuals during the entire generations in the GA inversion for Case 1. Fig. 4.9 presents that (i) the estimated values of parameters $P$ and $f$ of the entire individuals have wide ranges of values in the early generations; (ii) after the first 15 generations and until the 40-th generation, their values approach to their targeted values; and, (iii) lastly, their

Figure 4.6: The Doppler effect of the wave responses induced by a moving source in Example 1 considering the targeted material profile.
variations become significantly low during the last 10 generations. The histograms of the structural parameters in the three segments are shown in Fig. 4.10. It shows the excellent convergence of the estimated values of $E_2$ and $E_3$ in the first 20 generations. In contrast, the convergence of the estimated values of $E_1$ is much slower than those of $E_2$ and $E_3$. Nevertheless, our GA-based optimizer successfully updates the estimated value of $E_1$ such that it converges toward its targeted value with a very small value of the error ($\delta = 0.2\%$) as shown in Table 4.1. Lastly, Fig. 4.11 shows that the wave response, $u^m$, due to the targeted control parameters is in an excellent agreement with $u$ due to the reconstructed ones at two sensors located at $x$ of, respectively, 40 and 60 m.
Figure 4.8: The misfit functional for the best-fit individual over the GA iterations in Case 1.

Figure 4.9: The histograms of (a) $P$ and (b) $f$ of the entire individuals at all the generations in Case 1.
Figure 4.10: The histograms of (a) $E_1$, (b) $E_2$, and (c) $E_3$ of the entire individuals at all the generations in Case 1.
Figure 4.11: Wave responses, $u^m$ and $u$, at the sensors in Case 1.
4.4.2 Example 2 (Cases 6 to 9): joint inversion of two source parameters and nine stiffness parameters in a bridge comprised of nine piece wisely-homogeneous segments

This example considers a continuous beam, which consists of nine piece wisely-homogeneous segments (see Fig. 4.12). The source parameters and structural parameter values are estimated by our joint inversion solver. In this example, as the targeted stiffness parameter of the beam, \( E \) in the sixth segment \((1.8 \times 10^{10} \text{ Pa})\) is lower than those of the other segments \((2.5 \times 10^{10} \text{ Pa})\). Similarly to Example 1, the estimated value of \( E \) in each segment of the beam is bounded as \( 1.7 \times 10^{10} \leq E \leq 2.6 \times 10^{10} \text{ Pa} \) during the GA-based inversion simulation.

![Figure 4.12: A piece wisely-homogeneous Timoshenko beam with nine segments in Example 2.](image)

This example tests Cases 6 to 9, which are evaluated by using PS of 50, 100, 200, and 400, respectively, while all of them use NS of 45 with the sensor spacing of 2.2 m and GN of 50. Table 4.2 shows the reconstructed source parameter values, the error between their reconstructed and targeted values, and the average error of all the eleven control parameters in each case. The spatial distribution of the recovered stiffness of all the segments in each case is shown in Fig. 4.13. It presents that the discrepancy between the reconstructed and targeted values of stiffness parameters is decreased as PS is increased. Fig. 4.14 shows that the values of \( \mathcal{L} \) and \( \overline{E} \) for the best-fit individual in each case become smaller as the generation approaches to the last one. It also clearly shows the improvement of the accuracy of the joint inversion as we increase PS.
Figure 4.13: The reconstructed elastic modulus of a piece wisely-homogeneous beam of nine segments via the joint inversion in Example 2: (a) Case 6 using PS of 50, (b) Case 7 using PS of 100, (c) Case 8 using PS of 200, and (d) Case 9 using PS of 400.
Figure 4.14: (a) $\mathcal{L}$ and (b) $\overline{e}$ for the best-fit individual versus the GA iteration in Example 2.
Table 4.2: Example 2 - joint inversion of two source parameters and nine stiffness parameters in a bridge comprised of nine piece wisely-homogeneous segments by using NS of 45, GN of 50, and PS of different values: while only the source parameters are shown in this table, the stiffness parameters are visualized in Fig. 4.13.

<table>
<thead>
<tr>
<th>Cases</th>
<th>Source Parameters</th>
<th>Value</th>
<th>$\varepsilon$ (only source)</th>
<th>$\varepsilon$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Target</td>
<td>$P$ (N/m)</td>
<td>100</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>$f$ (Hz)</td>
<td>20</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Case 6</td>
<td>$P$ (N/m)</td>
<td>94.30</td>
<td>5.7%</td>
<td>5.2%</td>
</tr>
<tr>
<td>$PS = 50$</td>
<td>$f$ (Hz)</td>
<td>20.06</td>
<td>0.3%</td>
<td></td>
</tr>
<tr>
<td>Case 7</td>
<td>$P$ (N/m)</td>
<td>100.83</td>
<td>0.8%</td>
<td>3.2%</td>
</tr>
<tr>
<td>$PS = 100$</td>
<td>$f$ (Hz)</td>
<td>20.01</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td>Case 8</td>
<td>$P$ (N/m)</td>
<td>99.25</td>
<td>0.8%</td>
<td>2.9%</td>
</tr>
<tr>
<td>$PS = 200$</td>
<td>$f$ (Hz)</td>
<td>20.01</td>
<td>0.1%</td>
<td></td>
</tr>
<tr>
<td>Case 9</td>
<td>$P$ (N/m)</td>
<td>99.80</td>
<td>0.2%</td>
<td>1.9%</td>
</tr>
<tr>
<td>$PS = 400$</td>
<td>$f$ (Hz)</td>
<td>20.00</td>
<td>0.0%</td>
<td></td>
</tr>
</tbody>
</table>

4.5 Summary

We show the feasibility of simultaneously identifying the parameters of the stiffness distribution and a moving vibration source in a Timoshenko beam by using the presented GA-based inverse modeling. We tackle the inverse problem via the minimization of a misfit functional, which is calculated as the difference between sparsely-measured responses induced by target control parameters and computed counterparts due to estimated parameters.

The numerical results suggest the following findings. First, as shown in Example 1, the more sensors are deployed on the beam, the better accuracy of the presented joint inversion is obtained. Second, as shown in Example 2, in order to successfully invert for the material properties of a beam with a large number of segments, a large value of PS should be used. Both Example 1 (i.e., Cases 1 vs 2) and Example 2 (i.e., Cases 6 to 9) show that a larger value of PS leads to a better convergence of estimated parameters toward their targeted parameters, while the computational cost of the entire GA process is proportional to the multiplication between GN and PS.
CHAPTER 5
CONCLUSIONS

5.1 Summary of the dissertation
This dissertation explores a new computational method for accurately identifying complex, incoherent seismic input motion in a near-surface domain, truncated by a WABC for the first time. This dissertation also investigates the feasibility of conducting the simultaneous identification of both the material property of a Timoshenko beam and moving vibration source. Notably, we were able to demonstrate via numerical experiments that our newly-developed algorithms lead to the reconstruction of the targeted parameters.

We summarize the contributions of this research in the following.

- We found that the more heterogeneous the material property of a domain is, the larger the discrepancy between reconstructed and targeted traction is. It has been shown that this is due to the wave reverberation inside the numerical domain.

- We discovered that the OTD and DTO methods lead to the same inversion results in the presented examples shown in Chapter 2.

- The numerical minimizer suffers from solution multiplicity less when it identifies dynamic traction of lower frequency content than that of higher frequency content.
The presented robust numerical method can reconstruct dynamic traction that mimics inclined, incident plane waves of a complex, realistic earthquake signal in a domain truncated by a WABC.

Using a vertical array of sensors along with those on the top surface significantly improves the performance of our inversion solver compared to the case when sensors are deployed only on the top surface of the domain truncated by a WABC.

The more sensors are deployed on the beam, the better precision of the presented joint inversion is achieved.

To successfully invert for the source profile of a moving wave source and the material properties of a beam with a large number of segments by using the GA algorithm, a significant value of population size should be used.

5.2 Future extensions

To fulfill the gap between our numerical problem settings and those for a realistic application, we will extent the numerical and computational modeling, presented in chapters 2 and 3, as follows: the Domain Reduction Method (DRM) will be featured in a forward wave solver. Bielak and Christiano [55] and Bielak et al. [51] had developed the DRM, by which free-field wave motions are applied, as a dynamic input, along a fictitious boundary (also known as a DRM boundary) enclosed by the WABC. The DRM has been widely used for modeling wave behaviors of truncated solid domains subject to remote seismic excitations [56, 57, 58, 59, 60, 61].

Thus, the extension of the presented method will be aiming at reconstructing free-field seismic input motions at a DRM boundary. That is, we will spatially and temporally discretize estimated incident seismic wavefield functions at the two boundary surfaces of a single-element DRM buffer layer of the domain. For instance, we will discretize an $x$-component incident-wavefield function
at the horizontal boundary of a DRM buffer layer, i.e., \( u_{b_0}^0(x,y,t) \) or \( u_{e_0}^0(x,y,t) \)—the subscripts \( b \) and \( e \) denote the boundaries of the DRM layer neighboring the interior and exterior domains, respectively. Next, we will reconstruct the spatial and temporal distributions of the estimated incident seismic wavefield functions.

In concerning to the presented joint inversion, we will extend it as follows. First, we will extend this 1D beam model into a 3D model so that 3D wave responses of a realistic, detailed bridge model will be taken into account for the inverse modeling. Second, by using the adjoint equation-based approach, we could identify a much larger number of control parameters than those presented in this work. That is, the material property of each element in the finite element mesh of the 3D model can be inverted for by using the adjoint equation-based material tomography [18]. Please note that the presented GA-based joint inversion method is limited to detecting the material properties of the segments of a piece wisely homogeneous beam model. Thus, the adjoint equation-based material tomography could show the inversion performance of a higher resolution than the presented GA-based inversion method. At the same time, the profile of moving sources can be identified by using the adjoint equation-based source-reconstruction approach.

As shown in Chapters 2 and 3, the arbitrarily-varying spatial and temporal distributions of wave source functions can be identified by using the adjoint equation-based approach in the 1D and 2D scalar wave settings. As the advantage of the potential adjoint equation-based joint inversion method, due to its semi-analytical nature, its computational cost is small and does not depend on the number of control parameters that are to be identified.
BIBLIOGRAPHY


